MATHEMATICAL QUESTIONS

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

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Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6475. (By the late Prof. CLIFFORD, F.R.S.)—Let U, V be any two cubic functions of x; show that a quartic function $f(x)$ may always be found such that, by the substitution $y = \mathrm{U} : \mathrm{V}$, the elliptic differential $dx : \{f(x)\}^{\frac{1}{2}}$ will be transformed into $\mathrm{M}\,dy : \{\phi(y)\}^{\frac{1}{2}}$, where $\phi(y)$ is a quartic function of y, and M a constant.

Solution by ROBERT RAWSON.

Let $y = \dfrac{\mathrm{U}}{\mathrm{V}}$; then $\mathrm{V}^2 dy = \left(\mathrm{V}\dfrac{d\mathrm{U}}{dx} - \mathrm{U}\dfrac{d\mathrm{V}}{dx}\right) dx = \mathrm{D}\psi(x)\,dx$ ...... (1, 2).

But $$\frac{dx}{\{f(x)\}^{\frac{1}{2}}} = \frac{\mathrm{D}_1 \mathrm{V}^2 \phi(x)}{\{\mathrm{D}_1^2 \phi(x)^2 \,.\, x-\alpha_1 \,.\, x-\alpha_2 \,.\, x-\alpha_3 \,.\, x-\alpha_4\}^{\frac{1}{2}}} \cdot \frac{dy}{\mathrm{D}\psi(x)},$$

$$\therefore\quad \frac{dx}{(x-\alpha_1 \,.\, x-\alpha_2 \,.\, x-\alpha_3 \,.\, x-\alpha_4)^{\frac{1}{2}}} = \frac{\mathrm{D}_1\,dy}{\mathrm{D}\,(y-\beta_1 \,.\, y-\beta_2 \,.\, y-\beta_3 \,.\, y-\beta_4)} \text{...(3)},$$

if $\phi(x) = \psi(x)$ and $\mathrm{D}_1^2 \phi(x)^2 . x-\alpha_1 \ldots x-\alpha_4 = \mathrm{U}-\beta_1\mathrm{V} \ldots \mathrm{U}-\beta_4\mathrm{V}$... (4, 5).

It remains, therefore, to satisfy the equations (4) and (5), for any given rational functions of U and V in order that equation (3) may exist.

The function $\psi(x)$ is of $(2n-2)$ dimensions in x, when U and V are of (n) dimensions in x (See CAYLEY'S *Elliptic Functions*, page 163). By the Question, U and V are cubic functions.

Putting (6), $y = \dfrac{\mathrm{A}\,(x^3-px^2+qx-r)}{x^3-p_1x^2+q_1x-r_1}$, we have

$$\psi(x) = x^4 - \frac{2(q-q_1)}{p-p_1}x^3 + \frac{3r-3r_1+p_1q-pq_1}{p-p_1}x^2 - \frac{2(p_1r-pr_1)}{p-p_1}x + \frac{q_1r-qr_1}{p-p_1} \text{............(7)},$$

$$= x-\nu_1 \,.\, x-\nu_2 \,.\, x-\nu_3 \,.\, x-\nu_4 = \phi(x),$$

where $\nu_1, \nu_2, \nu_3, \nu_4$ are the roots of (7) and $\mathrm{D} = p-p_1$.

Equation (5) then becomes

$$\mathrm{D}_1^2 (x-\nu_1)^2 \ldots (x-\nu_4)^2 \,.\, x-\alpha_1 \ldots x-\alpha_4 = [\mathrm{A}\,(x^3-px^2+qx-r) - \beta_1(x^3-p_1x^2+q_1x-r_1)] \ldots [\mathrm{A}\,(x^3-px^2+qx-r) - \beta_4(x^3-p_1x^2+q_1x-r_1)] \text{...(8)},$$

or
$$(x-\nu_1)^2 \dots (x-\nu_4)^2 \,.\, x-\alpha_1 \dots x-\alpha_4$$
$$= \left(x^3 - \frac{Ap-\beta_1 p_1}{A-\beta_1}x^2 + \frac{Aq-\beta_1 q_1}{A-\beta_1}x - \frac{Ar-\beta_1 r_1}{A-\beta_1}\right)\dots$$
$$\dots\left(x^3 - \frac{Ap-\beta_4 p_1}{A-\beta_4}x^2 + \frac{Aq-\beta_4 q_1}{A-\beta_4}x - \frac{Ar-\beta_4 r_1}{A-\beta_4}\right) \quad \dots(9),$$
where
$$D_1^2 = A-\beta_1 \,.\, A-\beta_2 \,.\, A-\beta_3 \,.\, A-\beta_4 \quad \dots\dots\dots\dots(10).$$

Equation (9) may be transformed as follows by the relation
$$x^3 - ax^2 + bx - c = \left\{x - \tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 . \left\{x - \tfrac{1}{3}\left[a - 2(a^2-3b)^{\frac{1}{2}}\right]\right\} \dots(11),$$
if
$$\left\{\tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 . \left\{\tfrac{1}{3}\left[a - 2(a^2-3b)^{\frac{1}{2}}\right]\right\} = c \dots\dots\dots\dots(12).$$

Putting
$$\left.\begin{array}{l} a = \dfrac{Ap-\beta_1 p_1}{A-\beta_1}, \quad b = \dfrac{Aq-\beta_1 q_1}{A-\beta_1}, \quad c = \dfrac{Ar-\beta_1 r}{A-\beta_1} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_3 = \dfrac{Ap-\beta_4 p_1}{A-\beta_4}, \quad b_3 = \dfrac{Aq-\beta_4 q_1}{A-\beta_4}, \quad c_3 = \dfrac{Ar-\beta_4 r}{A-\beta_4} \end{array}\right\} \dots\dots\dots(13),$$

(9) becomes
$$(x-\nu_1)^2 \dots (x-\nu_4)^2 \,.\, x-\alpha_1 \dots x-\alpha_4$$
$$\left.\begin{array}{l} = \left\{x - \tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 \left\{x - \tfrac{1}{3}\left[a - 2(a^2-3b)^{\frac{1}{2}}\right]\right\} \dots \\ \dots \left\{x - \tfrac{1}{3}\left[a_3 + (a_3^2-3b_3)^{\frac{1}{2}}\right]\right\}^2 \left\{x - \tfrac{1}{3}\left[a_3 - 2(a_3^2-3b_3)^{\frac{1}{2}}\right]\right\} \end{array}\right\} \dots(14),$$
with the four conditional equations
$$\left\{\tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right]\right\}^2 \left\{\tfrac{1}{3}\left[a - 2(a^2-3b)^{\frac{1}{2}}\right]\right\} = c, \dots,$$
$$\left\{\tfrac{1}{3}\left[a_3 + (a_3^2-3b_3)^{\frac{1}{2}}\right]\right\} \left\{\tfrac{1}{3}\left[a_3 - 2(a_3^2-3b_3)^{\frac{1}{2}}\right]\right\} = c_3 \dots\dots\dots(15).$$

From (14), it readily follows that
$$\nu_1 = \tfrac{1}{3}\left[a + (a^2-3b)^{\frac{1}{2}}\right], \dots \nu_4 = \tfrac{1}{3}\left[a_3 + (a_3^2-3b_3)^{\frac{1}{2}}\right],$$
and
$$\alpha_1 = \tfrac{1}{3}\left[a - 2(a^2-3b)^{\frac{1}{2}}\right], \dots \alpha_4 = \tfrac{1}{3}\left[a_3 - 2(a_3^2-3b_3)^{\frac{1}{2}}\right] \dots\dots(16).$$

There are twelve equations—four in (15) and eight in (16)—to be satisfied by the quantities A, p, q, r, p_1, q_1, r_1, $\alpha_1 \dots \alpha_4$, $\beta_1 \dots \beta_4$, leaving only three arbitrary constants out of the fifteen in order to establish the differential equation
$$\frac{dx}{(x-\alpha_1 \dots x-\alpha_4)^{\frac{1}{2}}} = \frac{(A-\beta_1 \dots A-\beta_4)^{\frac{1}{2}}\, dy}{(p-p_1)\,(y-\beta_1 \dots y-\beta_4)^{\frac{1}{2}}} \quad \dots\dots\dots\dots(17).$$

To reduce (17) to the standard form, it is necessary to put $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = \dfrac{1}{k}$, $\alpha_4 = -\dfrac{1}{k}$, $\beta_1 = 1$, $\beta_2 = -1$, $\beta_3 = \dfrac{1}{k'}$, $\beta_4 = -\dfrac{1}{k'}$.

The systems (15), (16) may be written as follows:—
$$V_1^2 \alpha_1 = c, \ \dots \ V_4^2 \alpha_4 = c_3 \quad \dots\dots\dots\dots\dots\dots\dots\dots(18),$$
$$2V_1 = p - \alpha_1 \dots 2V_4 = p - \alpha_4, \ 3\alpha_1 = a - 2(a^2-3b)^{\frac{1}{2}} \dots 3\alpha_4 = a_3 - 2(a_3^2-3b_3)^{\frac{1}{2}} \dots(19).$$

6534. (By Prof. Salmon, F.R.S.)—Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that $aU_1 + bU_2 + cU_3 + dU_4 + eU_5$ may be either a perfect square L^2 or the

product of two factors MN. Prove that the line L touches a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point.

Solution by J. J. WALKER, M.A.

The lines M, N being $lx+my+nz$, $l'x+\dots$, and f an indeterminate multiplier, the six equations

$$a_1a+a_2b+\dots+a_5e+ll'f=0, \qquad 2f_1a+2f_2b+\dots+(mn'+m'n)f=0,$$
$$b_1a+b_2b+\dots+b_5e+mm'f=0, \qquad 2g_1a+2g_2b+\dots+(nl'+ln')f=0,$$
$$c_1a+c_2b+\dots+c_5e+nn'f=0, \qquad 2h_1a+2h_2b+\dots+(lm'+ml')f=0$$

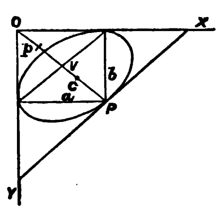
must hold. Eliminating $a \dots f$ among these, the result is

$$\mathrm{A}ll'+\mathrm{B}mm'+\mathrm{C}nn'+\mathrm{F}(mn'+nm')+\mathrm{G}(nl'+ln')+\mathrm{H}(lm'+ml')=0,$$

where A ... H are the minors of the determinant of the system of six equations formed by omitting the last column with the first, second ... sixth rows successively. But the relation among $l, m, n, l' \dots$ thus found is the condition that (MN) should be conjugate with respect to the conic $(\mathrm{BC}-\mathrm{F}^2)x^2+\dots+2(\mathrm{FG}-\mathrm{CH})xy=0$; so that, if M be given, N passes through the given point which is the pole of M with respect to this conic. As a particular case, if MN coincide in one line L, this must touch the same conic. In fact the relation then becomes $\mathrm{A}l^2+\dots+2\mathrm{H}lm=0$.

6440. (By Prof. TOWNSEND, F.R.S.)—The slab of the table in Question 6226 being supposed elliptical in place of rectangular; determine, all other particulars remaining unaltered, the slab of maximum area in its critical position, and the lengths of its semi-axes a_1 and b_1 in terms of the coordinates a and b of the post.

I. *Solution by* R. E. RILEY, B.A.; J. A. KEALY, M.A.; *and others.*

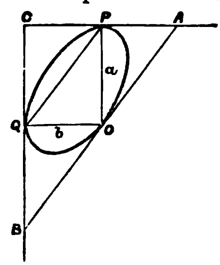


Going on the lines of the solution to Question 6226, we see that the ellipse must be the maximum ellipse inscribed in the triangle formed by the two lines and its tangent at the fixed point when in the critical position. Its area is therefore $\dfrac{2\pi ab}{3 \cdot 3^{\frac{1}{2}}}$. Also O is a point on the director circle, and CP = 2CV, OC = $\frac{2}{3}$OP; and therefore $a_1^2+b_1^2=\frac{4}{9}(a^2+b^2)$, and $a_1b_1=\dfrac{2ab}{3 \cdot 3^{\frac{1}{2}}}$, from which a_1 and b_1 can be found.

[This solution, though satisfactory so far as it goes, is incomplete, since the two normals a, b are not *always* the two *maxima* vectors of the ellipse from P.]

II. *Solution by the* PROPOSER.

Let CA, CB be the walls of the room, and O the position of the post; then, drawing through O the perpendiculars OP and OQ to the walls, and the parallel AB to the connector PQ of their feet P and Q, the ellipse OPQ, inscribed to the triangle ABC at the points of bisection O, P, Q of its sides, will, within certain limits of the ratio of $a:b$, be the contour of the required slab in its critical position.



For, the ellipse in question being that of greatest area that could be inscribed in the triangle ABC, no ellipse of greater area could under any circumstances attain to, much less pass through, the position in which its tangent at O would coincide with AB; and, that position being supposed attained in any manner by the ellipse OPQ, regarded as a rigid lamina movable in the plane of the figure; then, as, within certain limits of the ratio of $a:b$, OP and OQ are at once the two maxima vectors from O to the curve and the two minima vectors from O to the walls, a small rotation in either direction of the lamina round O as centre would, as in the case of the rectangular slab, disengage simultaneously its two points P and Q from contact with the walls, and extricate it consequently from its critical position.

To find the values of a_1 and b_1 in terms of a and b. Since, from the geometry of the ellipse, as may be shewn without difficulty,

$$a_1^2+b_1^2=\tfrac{4}{9}(a^2+b^2),\ \text{and}\ a_1^2\,b_1^2=\tfrac{4}{27}\,a^2b^2\,;$$

the values of a_1^2 and b_1^2 are consequently the roots of the quadratic equation

$$u^4-\tfrac{4}{9}(a^2+b^2)\,u^2+\tfrac{4}{27}\,a^2b^2=0,$$

and therefore $=\tfrac{2}{9}\left[(a^2+b^2)\pm(a^4+b^4-a^2b^2)^{\frac{1}{2}}\right]$, respectively.

To find the limits of the ratio of $a:b$ requisite to the validity of the above solution. The radii of curvature of the ellipse at P and Q, as may also be seen without difficulty, being respectively $\tfrac{1}{2}\dfrac{b^2}{a}$ and $\tfrac{1}{2}\dfrac{a^2}{b}$, in order that they should be less than a and b respectively, and therefore that OP and OQ should be both maxima vectors from O to the curve, we must have at once $b^2<2a^2$ and $a^2<2b^2$; and therefore, &c.

6485. (By ELIZABETH BLACKWOOD.)—Given that each of the compound events axy, byz, czx is always followed by two at least of the events d, e, f, and that each of the compound non-occurrences $d'x'y'$, $e'y'z'$, $f'z'x'$ implies the non-occurrence of two at least of the events a, b, c; what conclusion may thence be drawn with reference to the occurrence or non-occurrence of the events a, b, c, d, e, f without mentioning the events x, y, z?

Solution by H. McColl, B.A.; W. B. Grove, B.A.; *and others.*

The premises are (1) $axy+byz+czx : de+ef+fd$, and

$$(2) \quad d'x'y'+e'y'z'+f'z'x' : a'b'+b'c'+c'a'.$$

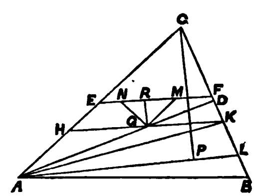
By mere inspection, we get $xyz : a'b'c'+de+ef+fd$,

$$x'y'z' : def+a'b'+b'c'+c'a', \quad x'yz : b'+de+ef+fd, \quad xy'z : c'+de+ef+fd,$$
$$xyz' : a'+de+ef+fd, \quad x'y'z : d+a'b'+b'c'+c'a', \quad xy'z' : e+a'b'+b'c'+c'a',$$
$$x'yz' : f+a'b'+b'c'+c'a'.$$

And the sum of all the antecedents implies the sum of all the consequents; that is, omitting redundant terms, $1 : a'+b'+c'+d+e+f$, which is equivalent to $abc : d+e+f$. This may be read, "If the events a, b, c all three happen, then one at least of the events d, e, f will happen.

5388 & 6471. (By Prof. Crofton, F.R.S.)—A triangle Δ is divided into two portions by a line joining two points taken at random within it; prove that the mean value (1) of the triangular portion is $\frac{4}{9}\Delta$; (2) of the portion that contains the centroid of the triangle is $\frac{\Delta}{3^6}(470+\frac{82}{3}\log 4)$; and (3) of the triangle formed by joining the centroid and the two points is $\frac{\Delta}{3^6}(34+\frac{16}{3}\log 2)$.

Solution by Professor Seitz, M.A.



Let ABC be the triangle, D the middle point of BC, G the centroid of the triangle, M and N two points such that EF, the line through them, meets AC and BC, and makes an angle with AC less than the angle A. Draw HGK and AL parallel to EF, and CP, GR perpendicular to AL, EF; and join AK. Let CP $= x$, EM $= y$, MN $= z$, EF$=y'$, area CAL $= u$, CEF $= v$, CHK $= v'$, and mean value throughout $=$ M.

1. We have AL $= 2ux^{-1}$, and $y' = 2u^{\frac{1}{2}}v^{\frac{1}{2}}x^{-1}$. An element of surface at M is $\frac{1}{2}xu^{-\frac{1}{2}}v^{-\frac{1}{2}}dv\,dy$, and at N it is $\frac{1}{2}x^2u^{-2}du\,z\,dz$. The limits of u are 0 and Δ, of v, 0 and u; of y, 0 and y'; and of z, 0 and y, and doubled. Hence, since the whole number of ways the points can be taken is Δ^2,

$$\frac{2}{\Delta^2}\int_0^{\Delta}\int_0^{u}\int_0^{y'}\int_0^{y}\frac{x^3}{4x^{\frac{1}{2}}}\,du\,v^{\frac{1}{2}}\,dv\,dy\,z\,dz = \frac{2}{3\Delta^2}\int_0^{\Delta}\int_0^{u}\frac{du}{u}\,v^2dv = \frac{2}{9\Delta^2}\int_0^{\Delta}u^2du = \frac{2}{27}\Delta.$$

This result is the same as that for each of five other expressions similar to that above. Hence the mean value M required is $\frac{4}{9}\Delta$.

2. We have $\quad$ area AKL $= \frac{2}{3}$ADL $= \frac{2}{3}(u-\frac{1}{2}\Delta)$,

ACK $= u-\frac{2}{3}(u-\frac{1}{2}\Delta) = \frac{1}{3}(u+\Delta)$; $\quad v' : \frac{1}{3}(u+\Delta) = \frac{1}{3}(u+\Delta) : u$, whence $v' = \frac{1}{9}(1+\Delta u^{-1})^2 u$.

When $u < \frac{1}{2}\Delta$, the portion ABFE contains the centroid for all values of v; when $u > \frac{1}{2}\Delta$, the portion ABFE contains the centroid from $v = 0$ to $v = v'$, and the portion CEF contains it from $v = v'$ to $v = u$. Hence

$$\frac{2}{\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}\int_0^{y'}\int_0^{y}\tfrac{1}{4}x^3(\Delta-v)\,u^{-\frac{5}{2}}\,du\,v^{-\frac{1}{2}}\,dv\,dy\,z\,dz$$

$$+\frac{2}{\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}(\Delta-v)\,v^{-\frac{1}{2}}\,dv+\int_{v'}^{u}v^{\frac{1}{2}}\,dv\right\}\int_0^{y'}\int_0^{y}\tfrac{1}{4}x^3u^{-\frac{5}{2}}\,du\,dy\,z\,dz$$

$$=\frac{2}{3\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}(\Delta-v)\,u^{-1}\,du\,v\,dv+\frac{2}{3\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}(\Delta-v)\,v\,dv+\int_{v'}^{u}v^2\,dv\right\}u^{-1}du$$

$$=\frac{1}{9\Delta^2}\int_0^{\frac{1}{2}\Delta}(3\Delta-2u)\,u\,du+\frac{1}{9\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{2u+\frac{\Delta}{3^3}\left(1+\frac{\Delta}{u}\right)^4-\frac{4u}{3^6}\left(1+\frac{\Delta}{u}\right)^6\right\}u\,du$$

$$=\frac{\Delta}{6\,.\,3^6}\left(470+\frac{82}{3}\log 4\right);$$

hence $$M=\frac{\Delta}{3^6}\left(470+\frac{82}{3}\log 4\right).$$

3. We have $GR = x^{-\frac{1}{2}}(v'^{\frac{1}{2}}-v^{\frac{1}{2}})$, or $xu^{-\frac{1}{2}}(v^{\frac{1}{2}}-v'^{\frac{1}{2}})$. When $u<\frac{1}{2}\Delta$, the area of the triangle MNG is $\frac{1}{2}xzu^{-}\ (v'^{\frac{1}{2}}-v^{\frac{1}{2}})$ for all values of v; when $u>\frac{1}{2}\Delta$, the area is the same as above from $v=0$ to $v=v'$; but from $v=v'$ to $v=u$, the area is $\frac{1}{2}xzu^{-\frac{1}{2}}(v^{\frac{1}{2}}-v'^{\frac{1}{2}})$. Hence we have

$$\frac{2}{\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}\int_0^{y'}\int_0^{y}\tfrac{1}{8}x^4(v'^{\frac{1}{2}}-v^{\frac{1}{2}})\,u^{-3}\,du\,v^{-\frac{1}{2}}\,dv\,dy\,z^2\,dz$$

$$+\frac{2}{\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}(v'^{\frac{1}{2}}-v^{\frac{1}{2}})\,v^{-\frac{1}{2}}\,dv+\int_{v'}^{u}(v^{\frac{1}{2}}-v'^{\frac{1}{2}})\,v^{-\frac{1}{2}}\,dv\right\}\int_0^{y'}\int_0^{y}\tfrac{1}{8}x^4u^{-3}\,du\,dy\,z^2\,dz$$

$$=\frac{1}{9\Delta^2}\int_0^{\frac{1}{2}\Delta}\int_0^{u}(1+\Delta u^{-1}-3u^{-\frac{1}{2}}v^{\frac{1}{2}})\,u^{-\frac{1}{2}}du\,v^{\frac{3}{2}}\,dv$$

$$+\frac{1}{9\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\int_0^{v'}\left(1+\frac{\Delta}{u}-\frac{3}{u^{\frac{1}{2}}}v^{\frac{1}{2}}\right)v^{\frac{3}{2}}dv+\int_{v'}^{u}\left(\frac{3}{u^{\frac{1}{2}}}v^{\frac{1}{2}}-1-\frac{\Delta}{u}\right)v^{\frac{3}{2}}dv\right\}u^{-\frac{1}{2}}\,du$$

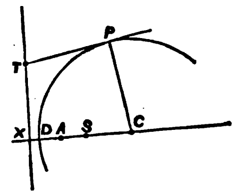
$$=\frac{1}{45\Delta^2}\int_0^{\frac{1}{2}\Delta}(2\Delta-3u)\,u\,du+\frac{1}{45\Delta^2}\int_{\frac{1}{2}\Delta}^{\Delta}\left\{\frac{2}{3^6}(1+\Delta u^{-1})^6+3-2\Delta u^{-1}\right\}u^2\,du$$

$$=\frac{\Delta}{6\,.\,3^6}\left(34+\frac{16}{3}\log 2\right);$$

hence we have $$M=\frac{\Delta}{3^6}\left(34+\frac{16}{3}\log 2\right).$$

6509. (By Prof. GENESE, M.A.)—Prove (1) that the directrices of a conic are common chords of the conic and its director circle; (2) that they are also the radical axes of that circle and of point-circles at the foci.

Solution by CHARLOTTE A. SCOTT; BELLE EASTON; *and others.*



Let A, S, C, XT be vertex, focus, centre and directrix of the conic; and CD = radius of director circle, so that $CD^2 = AC^2 + BC^2$; then, taking any point T on the directrix, drawing TP tangent to director circle, and joining TS, we have

$$\begin{aligned} TP^2 &= TC^2 - CP^2 = TX^2 + XC^2 - AC^2 - CB^2 \\ &= TX^2 + XS^2 + SC^2 + 2SX.SC - AC^2 - BC^2 \\ &= TS^2 + 2SX.CS - 2BC^2. \end{aligned}$$

Now we have $\dfrac{BC^2}{AC} : SX = SA : AX = CS : CA$;

hence $SX.CS = BC^2$, therefore $TP = TS$;

that is, T is on the radical axis of the director circle and a point circle at S; hence the directrix is this radical axis.

Let the ellipse and its director circle intersect at P, and draw PN perpendicular to the axis major, so that PN is common chord; then, since P is on the circle of centre C and radius CD, and also on conic, we have

$$PN^2 = CD^2 - CN^2, \quad PN^2 = \frac{BC^2}{AC^2}(AC^2 - CN^2) = BC^2 - \frac{BC^2}{AC^2}CN^2,$$

therefore $$CD^2 - CN^2 = BC^2 - \frac{BC^2}{AC^2}.CN^2;$$

but $CD^2 = AC^2 + BC^2$, therefore $CN^2 = \dfrac{AC^4}{AC^2 - BC^2} = \dfrac{CA^4}{CS^2}$.

But $CX^2 : AC^2 = AC^2 : CS^2$, hence $CX^2 = \dfrac{CA^4}{CS^2}$, $\therefore$ $CN = CX$;

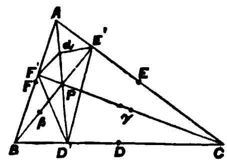
therefore the directrix is the common chord.

6504. (By Prof. TOWNSEND, F.R.S.)—A system of smooth equal spheres, two terminal and fixed, and the remainder in contact each with the two adjacent, being supposed to form an arch in a vertical plane, in unstable equilibrium under the action of gravity; determine, given all particulars, the ultimate form of the arch, when the component spheres are indefinitely increased in number and diminished in magnitude.

Solution by the Rev. A. L. WATHERSTON, M.A.; *the* PROPOSER; *and others.*

Every arc s of the curve of equilibrium of the system of spheres containing ultimately a number of spheres, and having, consequently, a mass proportional to its length, the several conditions for the equilibrium [unstable] of the system under the action of any forces are manifestly the same as for the equilibrium [stable] of a uniform flexible chord of the same length, mass, and terminal points, under the action of the same forces reversed in direction; and therefore, &c.

PROOF OF A NINE-POINT CIRCLE THEOREM. *By* C. PENDLEBURY, M.A.



Let the circle round D'E'F' cut PA at α; then

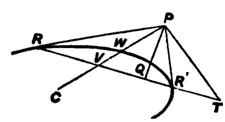
$\angle F'\alpha E' = \pi - \angle F'D'E' = \pi - (\pi - 2A) = 2A.$

But a circle will go round AF'PE', and PA is its diameter; hence α is its centre; therefore α, β, γ are the mid-points of PA, PB, PC.

[Mr. PENDLEBURY gives this proof as "much simpler and easier for boys to remember" than the proofs given in the usual text-books.]

5002. (By J. J. WALKER, M.A.)—Let S be a central conic, K its director circle, and S' the reciprocal polar of K with respect to S; if Q is the pole of the tangent to K at P, prove that PQ is normal to S'; and that the normal to S, at the point where it is met by the connector of P with the centre, is parallel to PQ.

Solution by CHRISTINE LADD; J. O'REGAN; *and others.*



Draw PR, PR' tangents from P to the conic S, and join CP, meeting RR' in V, and the conic in W. Then RR' is the tangent to the conic S' at Q; also, since RPR' is a right angle, VP = VR = VR'; and if PT, the tangent to the director circle, be drawn meeting RR' produced in T Q is the pole of PQ; therefore RR' is harmonically divided at Q and T; hence $VP^2 = VR'^2 = VQ \cdot VT$; therefore VQP, VPT are similar triangles, and VQP is a right angle. Hence PQ is normal to S' at Q; also, the tangent at W being obviously parallel to RR', the normal at W must be parallel to PQ.

SOLUTION OF A CUBIC EQUATION. *By* E. W. SYMONS, M.A.

Let the cubic be $\quad x^3+3ax+b=0$ ....(1);

then, assuming $x = y^{\frac{1}{3}}+z^{\frac{1}{3}}$, we have $x^3 \equiv y+z+3\,(yz)^{\frac{1}{3}}\,x$..............(2).

Making (1) and (2) identical, we have

$$y+z=-b, \quad yz=-a^3;$$

hence y and z are roots of the quadratic $u^2+bu-a^3=0$;

therefore $\quad x = \left(\dfrac{-b+\sqrt{b^2+4a^3}}{2}\right)^{\frac{1}{3}} + \left(\dfrac{-b-\sqrt{b^2+4a^3}}{2}\right).$

6560. (By W. B. Grove, B.A.)—In a certain school, every one who learns both English and French, or neither of them, does not learn Arithmetic; every one who learns Arithmetic learns both English and German or neither of them; everyone who learns French but not German learns either English or not Arithmetic. Assuming all combinations not inconsistent with these data to exist, prove (1) that the number of those who learn Arithmetic does not exceed the number of those who learn English; and (2) that the above conditions may be replaced by a single simple but exactly equivalent one.

Solution by H. McColl, B.A.; Elizabeth Blackwood; *and others.*

Speaking of some *one* boy, let a, e, f, g respectively denote the statements—he learns *Arithmetic*, he learns *English*, he learns *French*, he learns *German*. The data will then be expressed by the complex statement

$$(ef+e'f' : a')\,(a : eg+e'g')\,(fg' : e+a').$$

By inspection, we may reduce this to the form

$$a\,(ef+e'f'+e'g+eg'+e'fg') : 0.$$

Reducing the bracket factor in this implication to its *primitive form* (see my third paper in the *Proceedings of the Lond. Math. Soc.*), we get $a\,(e'+f+g') : 0$, that is, $a : ef'g$, as the simplest symbolical expression for the given conditions.

The number of boys to which the antecedent statement a is applicable is evidently an inferior limit to the number to which the consequent $ef'g$ is applicable, and, *a fortiori*, to the number to which e is applicable; that is to say, the number of boys that learn Arithmetic *cannot exceed* the number that learn English.

Note on the Solution of the 15-Puzzle in Question 6489 (*Reprint* Vol. XXXIV., p. 113).

By Rev. T. P. Kirkman, M.A., F.R.S.

My design was not simply to communicate the *rule* spoken of, which is well known, and has been given by Professor Tait in the Edinburgh Royal Society's *Proceedings*.

It appeared worth the while to show the connexion of the problem with the theory of substitutions. By whatever movement of the King S is reduced to S^0 = unity, as the natural order is called, the operation $S^{-1}S = 1$ has been performed by the unusual method of transpositions of an element undisturbed in S. In substitutions, $\phi\theta = \theta\phi$ is not generally true. It is true of $S^{-1}S = SS^{-1}$, because the product is a number. The results of the 13 operations (not upon S) indicated are

123856794	123476859	123875946
123896754	123479856	123895746
123876954	123475896	123845796
123976854	123975846	123849756

and $123846759 = S^2 = S^{3-1} = S^{-1}$.

The first, third, &c. are positive, none having an odd number of even circular factors. The second, fourth, &c. have each an odd number, and are negative. The second has one circle of 4, the fourth one of 2, the eighth one of 6. The terms positive and negative refer to the signs in the expansion of a determinant.

Unity being 123 ... 0*a* ... *f* of 16 elements,

769*b*310*ca*8*e*4*d*25*f* and 314690*ce*8*ba*572*df*

are soluble, having for even circles 8, 2 and 12, 2,

234567890*abcd*1*ef* and 2345619*a*0*ec*8*bd*7*f*

are insoluble, having one, the even circle 14 only, and the other 6, 4, 4. S has none, *i.e.*, an even number of even circles.

[On pp. 58—68 and 187—192 of Vol. I. of the *Messenger of Mathematics* there are some hints on substitutions, by Mr. Kirkman.]

6516. (By C. W. Merrifield, F.R.S.) — Prove that the remainders obtained by "casting out the nines" from the terms of any geometrical expression recur in regular order.

Solution by W. B. Grove, B.A.; Prof. Matz, M.A.; *and others.*

Let $a + ar + ar^2 + \ldots$ be the geometrical series, $a = 9m + n$, and $r = 9p + q$;

then $a + ar + ar^2 + \ldots = (9m + n) + (9m' + qn) + (9m'' + q^2 n) + \ldots$

Therefore the remainders are $n\,(1 + q + q^2 + \ldots)$; that is, the property in question will be true for all geometrical series, if it is true for all those which have the first term unity, and the common ratio any integer less than 9. On trial, this is found to be so for 1, 2, 4, 5, 7, 8, but not for 3, 6, or 9.

Therefore the theorem is true for all series in which the common ratio is not 3, 6, or 9.

6482. (By the Editor.)—Within a tetrahedron ABCD a point O is taken, and straight lines AOE, BOF, COG, DOH are drawn to meet the faces in E, F, G, H; find the average volumes of the tetrahedron EFGH.

Solution by Professor Seitz, M.A.

Let vol. OBCD $= w$, vol. OCDA $= x$, vol. ODAB $= y$, vol. OABC $= z$, vol. ABCD $= 1$; then we have $w + x + y + z = 1$, and

vol. OEFG : vol. OABC = OE . OF . OG : OA . OB . OC; whence

$$\text{vol. OEFG} = \frac{wxyz}{(1-w)(1-x)(1-y)} = \frac{wx}{(1-w)(1-x)}\left\{w+x+y-\frac{w+x}{1-y}\right\} = \text{V}.$$

The limits of w are 0 and 1; of x, 0 and $1-w$; and of y, 0 and $1-w-x$; hence the average volume of the tetrahedron OEFG is

$$\int_0^1\int_0^{1-w}\int_0^{1-w-x} \text{V}\, dw\,dx\,dy \div \int_0^1\int_0^{1-w}\int_0^{1-w-x} dw\,dx\,dy$$

$$= 6\int_0^1\int_0^{1-w}\int_0^{1-w-x} \frac{wx}{(1-w)(1-x)}\left\{w+x+y-\frac{w+x}{1-y}\right\} dw\,dx\,dy$$

$$= 3\int_0^1\int_0^{1-w} \frac{w}{1-w}\left\{2w+x+(w+x)^2-\frac{w(2+w)}{1-x}\right.$$
$$\left.-2\left(1+w+x-\frac{1+w}{1-x}\right)\log(w+x)\right\} dw\,dx$$

$$= \int_0^1 \left\{10w+7w^2+w^3+\frac{6w^2(2+w)}{1-w}\log w-\frac{6w(1+w)}{1-w}\log w\log(1+w)\right.$$
$$-\frac{6w(1+w)}{1-w}\left[\frac{1}{1+w}+\frac{1}{2^2}\cdot\frac{1}{(1+w)^2}+\frac{1}{3^2}\cdot\frac{1}{(1+w)^3}+\dots\right]$$
$$\left.+\frac{6w(1+w)}{1-w}\left[\frac{w}{1+w}+\frac{1}{2^2}\cdot\frac{w^2}{(1+w)^2}+\frac{1}{3^2}\cdot\frac{w^3}{(1+w)^3}+\dots\right]\right\} dw$$

$$= 50\tfrac{1}{4}-4\tfrac{1}{2}\pi^2-12\int_0^1\left\{\frac{1}{1+w}\log w\log(1-w)+\frac{1}{1-w}\log w\log(1+w)\right.$$
$$\left.-\frac{1}{w}\log(1-w)\log(1+w)\right\} dw$$

$$= 50\tfrac{1}{4}-4\tfrac{1}{2}\pi^2+12\left\{\log w\log(1-w)\log(1+w)\right\}_0^1$$
$$-24\int_0^1 \frac{dw}{1+w}\log w\log(1-w)$$

$$= 50\tfrac{1}{4}-4\tfrac{1}{2}\pi^2+24\int_0^1\left\{w-(1-\tfrac{1}{2})w^2+(1-\tfrac{1}{2}+\tfrac{1}{3})w^3\right.$$
$$\left.-(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4})w^4+\dots\right\}\log w\,dw$$

$$= 50\tfrac{1}{4}-4\tfrac{1}{2}\pi^2-24\text{S},$$

where $\text{S} \equiv \left\{\frac{1}{2^2}-(1-\tfrac{1}{2})\frac{1}{3^2}+(1-\tfrac{1}{2}+\tfrac{1}{3})\frac{1}{4^2}-(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4})\frac{1}{5^2}+\dots\right\}$.

The average volume of each of the tetrahedrons OFGH, OGHE, OHEF is the same as that of OEFG; therefore the average volume of the tetrahedron EFGH is $201-18\pi^2-96\text{S}$; and as the value of the series S, taken to 50 terms, is ·243113, the average volume required is ·0084. [The analogous average for a *triangle* has (*Reprint*, Vol. IX., pp. 49—52, Quest. 2537) been shown to be $10-\pi^2$, or ·1304. The evaluation, in definite terms, if possible, of the series S, or of the integral that leads to it, has been proposed as Question 6608.]

6532. (By Professor SYLVESTER, F.R.S.)—Prove that the condition of an equation of the 5th degree being linearly transformable into a recurring form is $I=0$, where, $\alpha, \beta, \gamma, \delta, \epsilon$ being the 5 roots, I is the product of the 15 distinct factors obtainable from the permutations of

$$\pm\left[(\alpha-\beta)(\alpha-\epsilon)(\delta-\gamma)+(\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)\right],$$

and show that I is a symmetrical function of the 18th order in the coefficients of the given equation.

[The question may be otherwise stated as follows:—From the system

$$\frac{pm+q}{rm+s}=\alpha,\quad \frac{p\mu+q}{r\mu+s}=\gamma,\quad \frac{p+qm}{r+sm}=\beta,\quad \frac{p+q\mu}{r+s\mu}=\delta,\quad \frac{p+q}{r+s}=\epsilon,$$

eliminate p, q, r, s, m, μ, so as to obtain the relation between $\alpha, \beta, \gamma, \delta, \epsilon$ in an integral form.]

Solution by W. J. C. SHARP, M.A.

The relations given in the note lead to

$$p^2-q^2-(pr-qs)(\alpha+\beta)+(r^2-s^2)\,\alpha\beta=0,$$
$$p^2-q^2-(pr-qs)(\gamma+\delta)+(r^2-s^2)\,\gamma\delta=0,$$
$$p^2-q^2-2(pr-qs)\,\epsilon+(r^2-s^2)\,\epsilon^2=0,$$

which involve $(\epsilon-\beta)(\epsilon-\delta)(\gamma-\alpha)+(\epsilon-\alpha)(\epsilon-\gamma)(\delta-\beta)=0$, the relation given in the question; consequently the condition required will be the product of the fifteen distinct factors of this form which can be formed. It will therefore be of order $18\ (3\times2+12)$, and weight 45, and being a function of the differences will satisfy the known differential equation; it is in fact, as stated in the question, the invariant I mentioned in SALMON'S *Higher Algebra*, p. 189, and which Dr. SALMON has calculated. (*Phil. Trans.* for 1858, p. 455.)

[That this is correct follows from the fact that, if $(a\,.\,b\ldots f)(x,1)^5=0$ be a reciprocal equation, it has one root (±1) in common with its Canonizant, and therefore the eliminant I vanishes.]

6549. (By the Rev. W. A. WHITWORTH, M.A.)—If $(3N)!$ be divided by ϵ (the base of Napierian logarithms), prove that the integer nearest to the quotient will be a multiple of $3N-1$.

Solution by W. H. BLYTHE, B.A.; J. H. TURRELL, M.A.; *and others.*

$$e^{-1}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\&c.+(-1)^N\left\{\frac{1}{(3N-2)!}+\frac{1}{(3N-1)!}+\frac{1}{(3N)!}-\&c.\right\}$$

$$\therefore\ (3N)!\div e=3\,.\,4\ldots(3N-1)\,.\,3N-4\,.\,5\ldots(3N-1)\,.\,3N+\ldots$$
$$\ldots(-1)^N\left\{3N(3N-1)-3N+1-\frac{1}{3N+1}+\&c.\right\}$$

$$=\text{a multiple of }(3N-1)+(-1)^N\left\{-\frac{1}{3N+1}+\frac{1}{(3N+1)(3N+2)}-\&c.\right\}.$$

Now this latter series is negative and less than $\frac{1}{3N+1}$, that is, less than $\frac{1}{2}$; thus the nearest integer to $(3N)! \div e$ is a multiple of $(3N-1)$.

[Mr. BLYTHE remarks that it appears to him that $N! \div e$ is a multiple of $(N-1) \pm$ a small fraction, according as N is odd or even, provided N is positive. To this Mr. WHITWORTH assents, and adds that if, moveover, f denote a small fraction, we have $N! \div e \pm f = M(N-1)$, where M is an integer such that $M \mp 1$ is a multiple of $N(N-3)$, the upper signs being taken when N is even and the lower when N is odd.]

6546. (By Professor PURSER, M.A.)—If the equation of a surface be expressed in the form $p = F(\theta, \phi)$, where p is the perpendicular from a fixed point on the tangent plane, θ, ϕ the usual angles determining its position; show that the sum of the principal radii of curvature is given by $R + R' = \Delta . p$, where Δ represents the differential operator

$$\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2}{d\phi^2} + 2,$$

corresponding to the equation of LAPLACE's functions of Order 1. Hence, employing BOOLE's form of solutions, it appears that the general equations of surfaces having their principal curvatures equal and opposite is

$$p = \frac{d}{d\theta}\left[\sin\theta\, F_1(e^{i\phi}\tan\tfrac{1}{2}\theta) + \sin\theta\, F_2(e^{-i\phi}\tan\tfrac{1}{2}\theta)\right].$$

Solution by J. LARMOR, M.A.; Prof. SCHEFFER, M.A.; *and others.*

Consider the directions of the two lines of curvature through the point. The radius of curvature of the section through each is obviously the same as that of a cross-section of the cylinder formed by the tangent planes along it, and is therefore $p + \frac{d^2p}{d\phi^2}$, with the usual notation. Thus the sum of the principal radii is $\left(\frac{d^2}{d\phi^2} + \frac{d^2}{d\psi^2} + 2\right)p$, where ϕ, ψ are the angles p makes with two fixed planes at right angles.

Now consider a unit sphere round the origin. The *flux* of a function V through an element of area bounded by the small circles $\phi, \phi + d\phi, \psi, \psi + d\psi$, is, as usual,

$$\left(\frac{d^2}{d\phi^2} + \frac{d^2}{d\psi^2}\right)V \,.\, d\phi\, d\psi,$$

therefore through any infinitesimal area is $\left(\frac{d^2}{d\phi^2} + \frac{d^2}{d\psi^2}\right)V \,.$ area. Now, take a polar element of area, and take θ, ϕ as polar coordinates; the flux through this area is

$$\frac{d}{d\phi}\left(\frac{dV}{\sin\theta \,.\, d\phi} \,.\, d\theta\right)d\phi + \frac{d}{d\theta}\left(\sin\theta \,.\, d\phi \,.\, \frac{dV}{d\theta}\right)d\theta,$$

which is same as $\left\{\frac{1}{1-\mu^2}\frac{d^2V}{d\phi^2} + \frac{d}{d\mu}(1-\mu^2)\frac{dV}{d\mu}\right\}.$ area, where $\mu = \cos\theta$.

Thus the result follows, and the solution as deduced by DONKIN in the *Phil. Trans.* for 1857. [Mr. LARMOR remarks that the Cartesian equation of the tangent plane may now be written down, and its envelop found; the envelop being the most general solution of the equation to a soap-film, (MONGE) $(1+x^2)\,r+2xys+(1+y^2)\,t=0$. In fact, if r^{-1} be put for p in Mr. PURSER's solution, we obtain the polar equation of the general solution of the equation derived from this by "Reciprocal Transformation" (corresponding to Duality, or Reciprocal Polars in Geometry), that equation being $(1+y^2)\,r-2xys+(1+x^2)\,t=0$.]

4350. (By Colonel CLARKE, C.B., F.R.S.)—Three points are taken at random in a given triangle; prove that the probability (p) that they enclose a given point whose triangular coordinates are (a, b, c), is

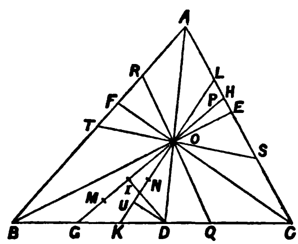
$$p = \mathrm{F}(a, b, c)+\mathrm{F}(b, c, a)+\mathrm{F}(c, a, b),$$

where $$\mathrm{F}(a, b, c) = 2abc\,(1-6a^2)+(6b^2c^2-24b^3c^3)\log\left(1+\frac{a}{bc}\right).$$

5234. (By S. WATSON.)—Three points are taken at random within a given triangle; prove that the chance that they will all lie on one side of some one line that can be drawn through the centroid of the triangle is

$$\frac{25}{27}-\frac{20}{81}\log 2.$$

Solution by Professor SEITZ, M.A.



(4350.) Let ABC be the given triangle, O the given point, and M, N, P the three random points. Produce AO, BO, CO to D, E, F. Let M be confined to one of the six triangles into which the given triangle is divided, as BOD; the results for the other triangles will be similar to the result thus obtained. Let N be taken, successively, in the triangles BOD, COD, COE, AOE, AOF, BOF. Let KL be the line through N when it is taken in BOD or AOE, QR the line through it when taken in COD or AOF, and ST the line through it when taken in COE or BOF. Draw GH through M, and draw DI, DU perpendicular to GH, KL.

Let $\mathrm{DI}=w$, DU or either of the perpendiculars from D, E, F on KL, QR, ST, $=x$, $\mathrm{OM}=y$, $\mathrm{ON}=z$, $\mathrm{OG}=y'$, $\mathrm{OK}=z'$, area $\mathrm{BOC}=a$, area $\mathrm{COA}=b$, area $\mathrm{AOB}=c$, area $\mathrm{ABC}=1$, area $\mathrm{ODG}=u$. Then we have

$$y'=2uw^{-1},\quad \text{area ODB}=u_1=\frac{ac}{1-a},\quad \text{area ODC}=u_2=\frac{ab}{1-a},$$

$$\text{area OEC}=u_3=\frac{ab}{1-b},\quad \text{area OEA}=u_4=\frac{bc}{1-b},\quad \text{area OFA}=u_5=\frac{bc}{1-c},$$

$$\text{area OFB}=u_6=\frac{ac}{1-c},\quad \text{and}\quad \text{area OEH}=u'=\frac{a^2b^2}{au_2+u}-bu_3;$$

and if v represents, successively, the areas ODK, ODQ, OES, OEL, OFR, OFT, we have area OEL $= v_1 = \dfrac{a^2b^2}{au_2+v} - bu_3$, area OFR $= v_2 = \dfrac{a^2c^2}{au_1+v} - cu_6$,

area OFT $= v_3 = \dfrac{b^2c^2}{bu_4+v} - cu_5$, area ODK $= v_4 = \dfrac{a^2b^2}{bu_3+v} - au_2$,

area ODQ $= v_5 = \dfrac{a^2c^2}{cu_6+v} - au_1$, area OES $= v_6 = \dfrac{b^2c^2}{cu_5+v} - bu_4$, and $z' = 2vx^{-1}$.

An element of the triangle at M is $\frac{1}{2}w^2u^{-2}\,du\,y\,dy$, and at N it is $\frac{1}{2}x^2v^{-2}\,dv\,z\,dz$. When N is in the triangle ODB, the number of favourable positions of P is represented by the area of OHL; when N is in the triangle ODC, the number of favourable positions is represented by the area of OHAR; and so on. Hence, since the whole number of ways the three points can be taken is expressed by unity, we have

$$\begin{aligned}
&\int_0^{u_1}\left\{\int_0^{u}(v_1-u')\,v^{-2}dv+\int_u^{u_1}(u'-v_1)\,v^{-2}dv+\int_0^{u_2}(u_4+u_5-u'-v_2)\,v^{-2}dv\right.\\
&\quad+\int_0^{u_3}(u_4+u_5-u'+v_3)\,v^{-2}dv+\int_0^{u'}(1-u_2-u_3-u'-v_4)\,v^{-2}dv\\
&\quad+\int_{u'}^{u_4}(u_2+u_3+u'+v_4)\,v^{-2}dv+\int_0^{u_5}(u_2+u_3+u'-v_5)\,v^{-2}dv\\
&\quad\left.+\int_0^{u_6}(u'+v_6)\,v^{-2}dv\right\}\int_0^{y'}\int_0^{z'}\tfrac{1}{4}w^2x^2u^{-2}\,du\,y\,dy\,z\,dz\\
=&\int_0^{u_1}\left\{\int_0^{u}(v_1-u')\,dv+\int_u^{u_1}(u'-v_1)\,dv+\int_0^{u_2}(u_4+u_5-u'-v_2)\,dv\right.\\
&\quad+\int_0^{u_3}(u_4+u_5-u'+v_3)\,dv+\int_0^{u'}(1-u_2-u_3-u'-v_4)\,dv\\
&\quad\left.+\int_{u'}^{u_4}(u_2+u_3+u'+v_4)\,dv+\int_0^{u_5}(u_2+u_3+u'-v_5)\,dv+\int_0^{u_6}(u'+v_6)\,dv\right\}du\\
=&\int_0^{u_1}\left\{2abc+4ab^2-6a^2b^2+\frac{2a^2b^2(1-4ab)}{au_2+u}-\frac{2a^4b^4}{(au_2+u)^2}+4a^2b^2\log\left(1+\frac{u}{au_2}\right)\right.\\
&\quad\left.-2a^2b^2\log\left(1+\frac{c}{ab}\right)-2a^2c^2\log\left(1+\frac{b}{ac}\right)+2b^2c^2\log\left(1+\frac{a}{bc}\right)\right\}du\\
=&\ 2a^2bc+10a^2b^2c-2a^2b^3-8abcu_2+2a^2b^2u_3+2a^2b^2(1-4ab)\log\left(1+\frac{c}{ab}\right)\\
&\quad+(4a^3b^2u_2+2a^2b^2u_1)\log\left(1+\frac{c}{ab}\right)-2a^2c^2u_1\log\left(1+\frac{b}{ac}\right)\\
&\quad+2b^2c^2u_1\log\left(1+\frac{a}{bc}\right).
\end{aligned}$$

Writing out the results for the remaining five triangles, and adding together the six results, we find the probability to be as stated in the question. [Another Solution is given on p. 63 of Vol. XXI. of the *Reprint.*]

(5234.) The chance that the three points lie on one side of some one line that can be drawn through O is $p_1 = 1-p$. When O is the centroid of the triangle, $a = b = c = \frac{1}{3}$, and $p_1 = \frac{25}{27} - \frac{80}{81}\log 2$.

Generalization of Prevost and Lhulier's Theorem in Chances.

By E. B. Elliott, M.A.

Statement.—A bag contains a number, known or unknown, of balls, of which we are only told that each one must be of one or other of n different colours, there being no presumption *à priori* that any one possible combination of the colours is more likely than any other. Of these balls $\Sigma(p)$ are now drawn, and found to be $p_1, p_2, p_3, \ldots p_n$ of the n colours severally, and are not replaced. To show that, if $\Sigma(\varpi)$ more be drawn, the chance that they be $\varpi_1, \varpi_2, \ldots \varpi_n$ of the colours respectively, is

$$\frac{\{\Sigma(\varpi)\}!\,\Pi\{(p+\varpi)!\}\,\{\Sigma(p)+n-1\}!}{\Pi(\varpi!)\,\Pi(p!)\,\{\Sigma(p+\varpi)+n-1\}!}.$$

Prevost and Lhulier's theorem (Todhunter's *History of Probability*, p. 454) is the particular case of this for two colours. For simplicity of expression, the statement has been made as to balls taken from a bag, but it will be at once clear that the same result applies to many other classes of observations.

Proof.—We should not have affected either the observation or the chance if, before drawing a ball, we had put our hand in the bag and fixed upon the $\Sigma(p)+1$ balls destined to be taken out in the first $\Sigma(p)+1$ drawings. Now, all possible hypotheses as to the combination of colours in the whole number of balls being *à priori* equally likely, it follows that all hypotheses possible with regard to $\Sigma(p)+1$ balls taken at random from among them are also equally likely. But of these various hypotheses only n are still tenable when of the $\Sigma(p)+1$ balls all but one have been taken up and found to be $p_1, p_2, \ldots p_n$ of the several colours; viz., the hypotheses that there were, of those colours respectively,

$$p_1+1, p_2, p_3, \ldots p_n;\ p_1, p_2+1, p_3, \ldots p_n;\ \ldots;\ p_1, p_2, p_3, \ldots p_n+1 \ \ldots (1, 2, 3).$$

Now, the chance of the observed event, on the supposition in turn of each of these hypotheses, is just that in each case that the single ball left to the $\Sigma(p)+1^{\text{th}}$ drawing be of the singular colour 1, 2, 3, ... or n. That is to say, the several chances are proportional to

$$p_1+1, p_2+1, p_3+1, \ldots p_n+1,$$

the common denominator being $\Sigma(p)+1$.

Hence the chances of the truth of the various hypotheses, estimated after the observed event, have the same numerators as these chances, their common denominator being the sum of those numerators, *i.e.* $\Sigma(p)+n$.

These are the chances after the $\Sigma(p)$ observed drawings, that the next one will give a ball of the first, second, ... n^{th} colour respectively.

Now, suppose this one drawing to be made, and that the resulting colour prove to be the r^{th}. Then the chances for the various colours at the next drawing are in like manner

$$p_1+1, p_2+1, \ldots p_r+2, \ldots p_n+1,$$

with common denominator $\Sigma(p)+n+1$.

Thus, returning to the moment after the first $\Sigma(p)$ drawings, the chance then that the next drawing will give colour r, and the one following colour s, is $\quad (p_r+1)(p_s+1) \div \{\Sigma(p)+n\}\{\Sigma(p)+n+1\}$,

unless $s=r$, in which case it is

$$(p_r+1)(p_r+2) \div \{\Sigma(p)+n\}\{\Sigma(p)+n+1\}.$$

We see also from the general form that the chances for the orders r, s and s, r are the same.

We may proceed now in this way to the consideration of any number we please of future drawings after the observed $\Sigma(p)$, and find at once that the chance in favour of the first $\Sigma(\varpi)$ drawings subsequent giving $\varpi_1, \varpi_2, \ldots \varpi_n$ of the various colours in any assigned order that we choose to fix upon, is

$$\frac{(p_1+1)(p_1+2)\ldots(p_1+\varpi_1).(p_2+1)\ldots(p_2+\varpi_2)\ldots.(p_n+1)\ldots(p_n+\varpi_n)}{\{\Sigma(p)+n\}\{\Sigma(p)+n+1\}\ldots\{\Sigma(p)+\Sigma(\varpi)+n-1\}},$$

a result, quite independent of that particular order, which we may write

$$\frac{(p_1+\varpi_1)!\,(p_2+\varpi_2)!\ldots(p_n+\varpi_n)!\,\{(\Sigma p)+n-1\}!}{p_1!\,p_2!\ldots\ldots p_n!\,\{\Sigma(p+\varpi)+n-1\}!},$$

or
$$\frac{\Pi\{(p+\varpi)!\}\{\Sigma(p)+n-1\}!}{\Pi(p!)\{\Sigma(p+\varpi)+n-1\}!}.$$

To find the chance of the $\Sigma(\varpi)$ drawings giving the required colour combinations in some order or other, we must multiply this by the number of different possible orders, *i.e.*, by $\{\Sigma(\varpi)\}!\div\Pi(\varpi!)$. And doing this we get the value given in statement.

Test of Accuracy. — Putting $p_1 = p_2 = \ldots = p_n = 0$, and letting $\varpi_1+\varpi_2+\ldots+\varpi_n = m$, the whole number of balls in the bag, substitution in one result gives for the chance in the first place of any particular division into parcels of the n colours of all the balls $1 \div$ the whole number of ways in which these balls can be divided into n parcels, a result independent of that particular division. And this was the supposition we started from.

Particular Example. — Referring to Quests. 4549, 4566, 4578, 4579 (*Reprint*, Vol. XXXIII., p. 68), I agree with Mr. Tebay that there is no ambiguity in the questions, but submit that their only admissible meaning is the one which Mr. Sanderson and myself attached to them, and not the one on which he and Mr. Leudesdorf have based their solutions. It seems to me that the question which they have solved is this: "A bag contains mn balls, of which every one must be of one or other of m different colours, *all different possible combinations of those colours being, it is presumed, equally likely*. All but m are drawn at random, and found to be $n-1$ of each colour. Find &c. &c." Thus stated, the theorems are included in the above general one, and the forms of result written down by Mr. Leudesdorf (without proof for the general case) are immediate by substitution.

5248. (By Professor Wolstenholme, M.A.)—If O, A, B be three points in order on a straight line, and a point P move so that OP is always a mean proportional between AP, BP; prove that (1) there exists also another system of three points A′, O′, B′, in order on the same straight line, such that O′P is a mean proportional between A′P, B′P; (2) O, O′ divide AB and A′B harmonically, and

$$OA+OB : OA'+OB' = OA.OB : OA'.OB' = (OA+OB)^2 : AB^2;$$

(3) the four points A, B, A′, B′ are the axial foci of the curve, which is a circular cubic, and the circle on OO′ as diameter is one of the circles with

respect to which the curve is its own inverse passing through four impossible foci; and (4) the vector equations of the curve are, if $AP = r_1$, $BP = r_2$, $A'P = r_3$, $B'P = r_4$, and $4n = \dfrac{AB^2}{OA \,.\, OB}$,

$$2(n-1)^{\frac{1}{2}}r_2 + [2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_3 + [-2n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_4 = 0,$$

$$2(n-1)^{\frac{1}{2}}r_1 + [2n^{\frac{1}{2}} - (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_3 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_4 = 0,$$

$$[2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_1 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}]r_2 + 2(n+1)^{\frac{1}{2}}r_4 = 0,$$

$$[2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}]r_1 + [-2n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]r_2 + 2(n+1)^{\frac{1}{2}}r_3 = 0.$$

Solution by Prof. NASH, M.A.; J. HAMMOND, M.A.; *and others.*

Taking O as origin, $OA = a$, $OB = b$, $OO' = \alpha$, $OA' = \beta$, $OB' = \gamma$, we have to show that α, β, γ can be so determined that the equations (1) and (2) are identical, $(x^2+y^2)^2 = [(x-a)^2+y^2]\,[(x-b)^2+y^2]$ ....(1),

$$[(x-\alpha)^2+y^2]^2 = [(x-\beta)^2+y^2]\,[(x-\gamma)^2+y^2] \quad \text{............(2).}$$

The necessary conditions are

$$\frac{\beta+\gamma-2\alpha}{a+b} = \frac{\beta^2+\gamma^2-2\alpha^2}{a^2+b^2} = \frac{\beta\gamma-\alpha^2}{ab} = \frac{\beta\gamma\,(\beta+\gamma)-2\alpha^3}{ab\,(a+b)} = \frac{\beta^2\gamma^2-\alpha^4}{a^2b^2} = k.$$

From these equations, we obtain

$$\beta+\gamma+2\alpha = a+b, \qquad \beta\gamma+\alpha^2 = ab,$$

$$\beta+\gamma-2\alpha = k\,(a+b), \qquad \beta\gamma-\alpha^2 = kab;$$

therefore
$$\beta+\gamma = \tfrac{1}{2}(1+k)\,(a+b), \qquad \beta\gamma = \tfrac{1}{2}(1+k)\,ab,$$

$$2\alpha = \tfrac{1}{2}(1-k)\,(a+b), \qquad \alpha^2 = \tfrac{1}{2}(1-k)\,ab;$$

therefore $\tfrac{1}{2}(1+k) = \left(\dfrac{a-b}{a+b}\right)^2 = \dfrac{n}{n+1}$, $\tfrac{1}{2}(1-k) = \dfrac{4ab}{(a+b)^2} - \dfrac{1}{n+1}$;

therefore
$$\frac{\beta+\gamma}{a+b} = \frac{\beta\gamma}{ab} = \frac{(a-b)^2}{(a+b)^2} = \frac{n}{n+1};$$

i.e.,
$$\frac{OA+OB}{OA'+OB'} = \frac{OA\,.\,OB}{OA'\,.\,OB'} = \frac{(OA+OB)^2}{AB^2}.$$

Also
$$\frac{2}{OO'} = \frac{2}{\alpha} = \frac{8}{(1-k)\,(a+b)} = \frac{a+b}{ab} = \frac{1}{a}+\frac{1}{b} = \frac{1}{OA}+\frac{1}{OB}$$

$$= \frac{\beta+\gamma}{\beta\gamma} = \frac{1}{OA'}+\frac{1}{OB'};$$

therefore OO′ divides AB and A′B′ harmonically. Therefore A′, O′, B are in order.

The form of (1) shows that $y \pm i\,(x-a) = 0$ are tangents, therefore A is a focus, and similarly A′, B, B′ are foci.

Taking the middle point of OO′ as origin, the equation of the locus is

$$[2x\,(a+b)-(a-b)^2]\,(x^2+y^2) + \frac{2a^2b^2}{a+b}\,x = 0,$$

or
$$2r^2\cos\theta\,(a+b) - r\,(a-b)^2 + \frac{2a^2b^2}{a+b}\cos\theta = 0,$$

and this is evidently its own inverse with respect to the circle $r = \dfrac{ab}{a+b}$, *i.e.*, the circle on OO′ as diameter. It is proved, in the solution

of Quest. 5878, that if d_1, d_2, d_3, d_4 be the distances of the foci from the foot of the asymptote, the vector equations are of the form

$$(d_3^2-d_4^2)\,r_2+(d_4^2-d_2^2)\,r_3+(d_2^2-d_3^2)\,r_4=0.$$

In this case the asymptote is $x=\dfrac{a^2+b^2}{2(a+b)}=\dfrac{2n+1}{4(n+1)}(a+b)$, and the distances d_1, d_2, d_3, d_4 are proportional to

$$1\pm 2\,[n(n+1)]^{\frac{1}{2}},\quad -1\pm 2\,[n(n-1)]^{\frac{1}{2}}.$$

Hence the four vector equations can be found. These equations can also be found linearly by taking the distances of the vertices from the foci.

6551. (By R. A. Roberts, M.A.)—If three lines, making an angle θ with a conic at points whose eccentric angles are α, β, γ, meet in a point, show that $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=\dfrac{2ab}{a^2-b^2}\cot\theta$.

Solution by Prof. Wolstenholme, M.A.; E. Rutter; *and others.*

If OP, OQ, OR be the three lines, the triangle formed by other three straight lines drawn through P, Q, R at angles to OP, OQ, OR, will be determinate in form whatever ϕ may be, and any one of the sides will vary as $\sin\phi$. Hence, in the two triangles formed by the normals, and by the tangents, at P, Q, R, the ratio of homologous sides will be $\sin(\frac{1}{2}\pi-\theta):\sin\theta$, or $\cot\theta:1$. But this ratio is well known to be

$$\frac{a^2-b^2}{2ab}\left[\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\right]:1.$$

Hence, if two concentric conics be such that triangles PQR can be inscribed in one whose sides touch the other, straight lines can be drawn through P, Q, R to a point O such that OP, OQ, OR are inclined to the respective normals at P, Q, R at the same angle which remains constant for all the triangles.

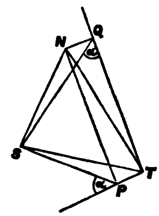
[If the triangles be respectively inscribed in and circumscribed to the ellipses $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $\mathrm{A}\dfrac{x^2}{a^2}+\mathrm{B}\dfrac{y^2}{b^2}+2\mathrm{H}\dfrac{xy}{ab}=1$, the angle will be

$$\tan^{-1}\left(\frac{a^2-b^2}{4ab}\cdot\frac{\mathrm{H}}{\mathrm{H}^2-\mathrm{AB}}\right).$$

A, B, H are connected by the equation $(\mathrm{A}+\mathrm{B}+\mathrm{H}^2-\mathrm{AB})^2=4(\mathrm{AB}-\mathrm{H}^2)$.]

6026. (By W. H. H. Hudson, M.A.)—If P, Q be two points on an equiangular spiral such that the tangents and normals thereat intersect at right angles in T, N respectively, prove that the locus of N is the evolute of the locus of T.

Solution by D. Edwardes; Prof. Nash, M.A.; *and others.*



By taking a consecutive position of the tangents, NT is seen to be the normal at T to the locus of T. Join SN, ST. A circle will evidently pass through SPTQN. Hence $\angle$ STN = $\angle$ SQN, and consequently the locus of T is a *similar* equiangular spiral. Again, $\angle$ SNT = $\angle$ SQT = the constant angle of spiral. But this is the construction for the centre of curvature of the equiangular spiral described by T. Hence the locus of N is the evolute of the locus of T.

6426. (By C. Leudesdorf, M.A.)—Prove that, if n be odd, the product of the squares of the differences of the roots of the equation

$$x^n + nx^{n-1} + (n-1)^{n-1} = 0 \quad \text{is} \quad 2n^n (n-1)^{(n-1)^2}.$$

Solution by the Rev. D. Thomas, M.A.; H. Hillary, M.A.; *and others.*

Let $a_1, a_2, \ldots a_n$ be roots of $f(x) = x^n + nx^{n-1} + (n-1)^{n-1} = 0$; then

$$(a_1-a_2)^2 (a_1-a_3)^2 \ldots = (-1)^n f'(a_1) f'(a_2) \ldots f'(a_n)$$
$$= -\left[na_1^{n-1} + n(n-1)\,a_1^{n-2}\right]\left[na_2^{n-1} + n(n-1)\,a_2^{n-2}\right] \ldots (n \text{ being odd})$$
$$= -n^n (a_1 a_2 \ldots a_n)^{n-2} (n-1+a_1)(n-1+a_2)(n-1+a_3) \ldots$$
$$= -n^n (n-1)^{(n-1)(n-2)} f\{-(n-1)\}$$
$$= n^n (n-1)^{(n-2)(n-1)} \{-(n-1)^n + n(n-1)^{n-1} + (n-1)^{n-1}\}$$
$$= n^n (n-1)^{(n-1)^2} [-(n-1) + n + 1] = 2n^n (n-1)^{(n-1)^2}.$$

4795 & 4198. (By Prof. Crofton, F.R.S.)—Prove that—(1) the arc of a Cartesian oval, at any point p, is equally inclined to the straight line from p through any one point, and to the circular arc from p through the other two foci; and (2), any system of triconfocal Cartesian ovals intersect each other orthogonally. [By triconfocal is meant having their three collinear foci common. Confocal conics furnish a particular case of this theorem, as they have a third common focus at an infinite distance.]

Solution by Professor Nash, M.A.

I assume the following theorem, which can be very easily proved:—If the constant of inversion be so chosen that a Cartesian is its own inverse with respect to one focus A, then the other two foci B, C are inverse

points; or, in another form, If a circle through BC meet the curve in P, and AP meet the circle again in P′, then P′ is also upon the Cartesian."

Now, taking two circles, centres A, B and radii a, b, and any point O in the line AB, if through O a line be drawn cutting the circles in Q, R and Q′, R′, the four intersections of the lines AQ, AR with the lines BQ′, BR′ all lie upon a Cartesian having A, B as foci.

Also, the tangent at any one of these points, and the corresponding tangents to the circles, are concurrent. (CHASLES's *Aperçu Historique*, p. 853.)

In the figure, considering OQQ′ as a transversal cutting the sides of the triangle PAB, we get at once the vector equation

$$\frac{\mathrm{OA}}{a}.\mathrm{AP}-\frac{\mathrm{OB}}{b}.\mathrm{BP}=\mathrm{AB}.$$

Now, consider the four points in which AQ meets the Cartesian; two of these, P, P′, are determined as above by the line OQ, and the other two by the line Oq, QAq being a diameter of the circle. Hence P, P′ are inverse points with respect to A, therefore the circle through PP′B passes through the third focus C.

If PK be the tangent at P to this circle, the angle KPA is equal to the angle PBP′; also, the angles PQT, PQ′T are right angles, therefore the angle QPT = angle QQ′T = ½ angle PBP′, therefore the angle KPA = double the angle TPQ.

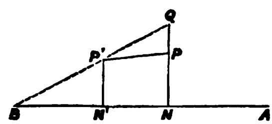
If the tangents at P, P′ meet in Y, the angles YPP′, YP′P are equal; therefore YP = YP′, and Y lies on the circle round BCPP′.

Hence also, if two Cartesians have the same foci, they will cut each other at right angles, which is Quest. 4198.

[For another solution of Quest. 4795, see *Reprint*, Vol. XXV., p. 17.]

6276. (By Prof. TANNER, M.A.)—In measuring the distance between two "stations," a surveyor, holding one end of the measuring chain at the near station, places his assistant, who has the other end of the chain, in a direct line between himself and the further station. The assistant leaves a mark from which the operation is repeated, and so on until the line is finished. Suppose that, instead of directing the assistant from the proper place, the surveyor always stands a small distance (h) to the right; find (1) the deviation produced after measuring n chains, the distance between the stations being m chains; and (2) the maximum deviation.

Solution by the PROPOSER.



In the figure, AB is the line to be measured, and P is the end of the n^{th} chain. PQ = h, PP′ = 1, PN = δ_n, P′N′ = δ_{n+1}. Also, very nearly, BN = $m-n$, BN′ = $m-n-1$ (all distances being in "chains").

From the figure, $$\frac{\delta_{n+1}}{m-n-1}=\frac{\delta_n+h}{m-n};$$

whence $$\delta_n=(m-n)\,h\left\{\frac{1}{m}+\frac{1}{m-1}+\ldots+\frac{1}{m-n+1}\right\}=m-n\,.\,h\,.\,\mathrm{S}_n,\ \text{say}.$$

Therefore $\delta_{n+1} - \delta = h(1 - S_{n+1})$.

Hence the necessary and sufficient conditions that δ_n is a maximum are $S_n < 1 < S_{n+1}$.

If $S_{n+1} = 1$, δ_n and δ_{n+1} have the same value, and each is a maximum.

[The determination of n from these data has been proposed for solution as a separate Question.]

6464. (By W. S. B. Woolhouse, F.R.A.S.)—Show that

$$\int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = 2 \cdot \frac{2.4 \ldots 2n}{3.5 \ldots 2n+1} (a+x)^{\frac{1}{2}} \left\{ a^n \left(1 + \frac{a}{x}\right)^{-\frac{1}{2}} \text{ as far as } x^n \right\}.$$

Solution by G. F. Walker, M.A.; R. Knowles, L.C.P.; *and others.*

$$\int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = 2x^n (a+x)^{\frac{1}{2}} - 2n \int \frac{x^{n-1}(a+x)\,dx}{(a+x)^{\frac{1}{2}}},$$

$$(2n+1) \int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = 2x^n (a+x)^{\frac{1}{2}} - 2na \int \frac{x^{n-1}\,dx}{(a+x)^{\frac{1}{2}}};$$

therefore $$\int \frac{x^n\,dx}{(a+x)^{\frac{1}{2}}} = \frac{2x^n (a+x)^{\frac{1}{2}}}{2n+1} - \frac{2na}{2n+1} \int \frac{x^{n-1}\,dx}{(a+x)^{\frac{1}{2}}} = \&c.;$$

and since coefficient of x^n in $\left(1+\frac{x}{a}\right)^{-\frac{1}{2}}$ is $\frac{(-1)^n}{a^n} \frac{1.3.5 \ldots 2n-1}{2^n (n)!}$, the result follows.

6522. (By W. R. Westropp Roberts, M.A.)—A sphere is gently placed on a rough inclined plane of given height and length, and descends under the influence of gravity; prove that the value of the coefficient of friction for which the *vis viva* gained in the descent is a minimum is

$$2mgh \times \frac{2a}{a + (a^2 + k^2)^{\frac{1}{2}}}.$$

Solution by G. S. Carr, B.A.

Let the acceleration of the centre of the sphere be f, and the angular acceleration F; then, with the usual notation,

$$f = g(\sin\alpha - \mu\cos\alpha), \quad F = \frac{a}{k^2}\mu g\cos\alpha,$$

both being constants. Therefore, t being the time of descent, $v = ft$, $\omega = Ft$, $v^2 = 2fh\operatorname{cosec}\alpha$. The *vis viva* acquired will be

$$V = m(v^2 + k^2\omega^2) = mv^2\left(1 + k^2\frac{F^2}{f^2}\right).$$

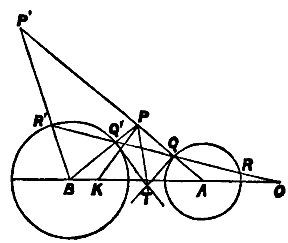
Substitute the values of F, f, v^2, and put $\mu = x \tan \alpha$, $k^2 = \frac{2}{5}a^2$; then

$$V = mgh\frac{2-4x+7x^2}{1-x}. \quad \text{Then } \frac{dV}{dx} = 0 \text{ gives } x = 1 \pm \sqrt{\tfrac{5}{7}}.$$

Since x must be $< \frac{2}{7}$, the least root is the one required. Substituting this value of x in V, we obtain the given result.

3930. (By Professor WOLSTENHOLME, M.A.) — In the limaçon $r = a(e + \cos\theta)$, if S be the pole, O the centre of inversion, OpP any chord through O, and y the point of intersection of the tangents at P, p, then y lies on the circle SpP, which touches the axis at S; Py = py, and the locus of y is the cissoid $r\cos\theta + a\sin^2\theta = 0$.

Solution by Prof. SCHEFFER, M.A.; Prof. NASH, M.A.; *and others.*



The first part of this is a particular case of the Corollary given in my solution of Quest. 4795, the two foci B, C coinciding at the pole of the limaçon, and the single focus A being the centre of inversion, or point, with respect to which the curve is its own inverse.

Inverting with respect to S, the proposition becomes this: Pp is a chord of a central conic parallel to the transverse axis, two circles pass through a focus S and touch the conic at P, p respectively; the second point (y) of intersection of these circles lies upon Pp; SP : Py = Sp : py; and the locus of y is a parabola having its vertex at S.

The point y lies on the line joining S to the pole of Pp; therefore Sy bisects $\angle$ PSp. The rest is easily proved.

6421. (By W. H. H. HUDSON, M.A.) — If a point P moves so that SP . PM is constant, where S is a fixed point and PM is perpendicular to a fixed line, and if tangents at any two points of its path be drawn and produced to meet the asymptote, prove that the triangle included between the two tangents and the asymptote is equal to the area included between them and the corresponding arc of the locus of the extremity of the polar subtangent.

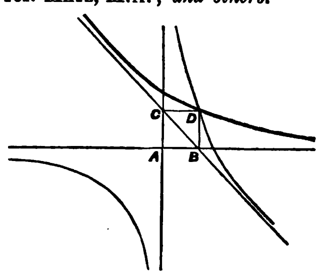
Solution by G. F. WALKER, M.A.; J. O'REGAN; *and others.*

It is a known property of this curve, that the part of the tangent intercepted between the extremity of the polar subtangent and the line which is the locus of M, is bisected at the point of contact.

Hence the whole area swept out by this line between the curve, tangents, and the line, will be equal to that between the curve, tangents, and the locus of the polar subtangent; and, since the part between the tangents and the curve will be common to both, the result follows.

5869. (By C. Leudesdorf, M.A.)—ABDC is a square. Find the equation, referred to AB, AC as axes, of the cubic curve having AB, BC, CA as asymptotes, and having a double point at D; and trace the curve completely.

Solution by Prof. Nash, M.A.; Prof. Matz, M.A.; *and others.*



The equation of a cubic having $x = 0$, $y = 0$, $x + y - a = 0$ for asymptotes, must be of the form

$$xy\,(x + y - a) + Ax + By + C = 0,$$

which, transformed to parallel axes through D, becomes

$$xy\,(x + y) + a\,(x^2 + 3xy + y^2) + x\,(A + 2a^2) + y\,(B + 2a^2) + Aa + Ba + C + a^3 = 0.$$

Since D is a double point, we have

$$A = -2a^2 = B,\ C = 3a^3,$$

and the equations referred to AB, AC and DB, DC are

$$xy\,(x + y - a) = a^2(2x + 2y - 3a),\quad xy\,(x + y) + a\,(x^2 + 3xy + y^2) = 0 \ldots (A, B).$$

From (B) we see that the approximate form of the curve at D is $y^2 + 3xy + x^2 = 0$, and the tangents to the two branches make angles $\tan^{-1} \frac{1}{2}(-3 \pm \sqrt{5})$ with the axis of x. The curve cuts the asymptotes $y = 0$, $x = 0$ at the points $x = \frac{3}{2}a$, $y = 0$, and $x = 0$, $y = \frac{3}{2}a$, but does not meet the third asymptote in any point at a finite distance from A. From (A),

$$\frac{dy}{dx} = \frac{2a^2 - y\,(x + y - a) - xy}{x\,(x + y - a) + xy - 2a^2} = -\tfrac{8}{5} \text{ when } x = \tfrac{3}{2}a,\ y = 0.$$

The curve is evidently symmetrical with respect to the line $x = y$, and meets it again in the point $x = y = -\frac{3}{2}a$; therefore the form is that shown in the figure.

6539. (By Prof. Minchin, M.A.)—Prove the following fundamental theorem :—In all motion of a rigid body parallel to one plane, there is at every instant a point which has no acceleration (let us call it the *instantaneous acceleration centre*, and denote it by J); the acceleration of every point in the body is directly proportional to its distance from J, and its

direction makes a constant angle with the line joining the point to J. [If P is any point in the body, and ω, $\dot{\omega}$ the angular velocity and angular acceleration of the body at any instant, the acceleration of P is $(\omega^4+\dot{\omega}^2)^{\frac{1}{2}}$. JP, and $\tan\phi = \frac{\dot{\omega}}{\omega^2}$, where ϕ = angle between JP and direction of acceleration. Hence there is construction for J analogous to that for I, the instantaneous velocity centre.]

I. *Solution by* G. F. WALKER, M.A.; Prof. MATZ, M.A.; *and others*.

Let $\ddot{x}$, $\ddot{y}$, $\ddot{\theta}$ be the accelerations of the centre of gravity, and the angular acceleration about it, and let (x, y) be the coordinates of any point of the body relative to the centre of gravity. The acceleration of this point will be (since its distance from the centre of gravity is constant),

$$\ddot{x}-x\dot{\theta}^2-\ddot{\theta}y,\quad \ddot{y}-y\dot{\theta}^2+\ddot{\theta}x \quad\text{.........................(1).}$$

If we choose x_1, y_1 so that $\ddot{x}-x_1\dot{\theta}^2-\ddot{\theta}y_1=0$, and $\ddot{y}-y_1\dot{\theta}^2+\ddot{\theta}x_1=0$, we get the position of J, the acceleration centre, and the expressions (1) become $\dot{\theta}^2(x_1-x)+\ddot{\theta}(y_1-y)$ and $\ddot{\theta}(x_1-x)-\dot{\theta}^2(y_1-y)$. Squaring and adding, we find $(\dot{\theta}^4+\ddot{\theta}^2)^{\frac{1}{2}}$JP for the complete acceleration, and the form of the expressions shows that the direction makes an angle $\tan^{-1}\frac{\ddot{\theta}}{\dot{\theta}^2}$ with JP.

II. *Solution by* J. J. WALKER, M.A.

Let O, O′ be the centres of instantaneous angular velocity and acceleration respectively. Describe a circumference on OO′ as diameter, and in OO′ take Q, so that OQ : QO′ = $\dot{\omega}^2 : \omega^4$. Draw J′QJ perpendicular to OO′ to meet the circumference in J′J, J being the one of the two points such that $\dot{\omega}$ is in the direction OJ. Then J will be the "instantaneous acceleration centre," since $\quad \text{OJ}\omega^2 = \text{O}'\text{J}\dot{\omega}$..................................(1).

Let P be any other point in the same plane. Its accelerations will be $\dot{\omega}$. O′P, perpendicular to O′P, and ω^2OP in the direction PO. Resolving these in the direction PJ, and in that of a perpendicular to PJ, the components of their resultant (a) in these directions will be

$$X=\omega^2\text{OP}\cos\text{OPJ}-\dot{\omega}\text{O}'\text{P}\sin\text{O}'\text{PJ}=\omega^2(\text{JP}+\text{OJ}\sin\text{O}'\text{JP})-\dot{\omega}\text{O}'\text{P}\sin\text{O}'\text{PJ}$$
$$=\omega^2\text{JP in virtue of (1); and}$$
$$V=\omega^2\text{OP}\sin\text{OPJ}+\dot{\omega}\text{O}'\text{P}\cos\text{O}'\text{PJ}=\omega^2\text{OJ}\cos\text{O}'\text{JP}+\dot{\omega}\text{O}'\text{J}\cos\text{O}'\text{PJ}\sin\text{O}'\text{JP}$$
$$+\sin\text{O}'\text{PJ}=\dot{\omega}\text{O}'\text{J}\sin\text{PO}'\text{J}+\sin\text{O}'\text{PJ by (1)}=\dot{\omega}.\text{JP}.$$

Hence, at once, $\qquad a=(\omega^4+\dot{\omega}^2)^{\frac{1}{2}}\text{JP},\ \tan\phi=\dot{\omega}\div\omega^2.$

[Mr. WALKER states that the foregoing solution was written out before he had seen Prof. EVERETT's reference to SCHELL's *Memoir* (1873) as having anticipated Prof. MINCHIN in the discovery of the "Centre of Acceleration." To SCHELL's acknowledgment of the priority of BRESSE (1853) in noticing the point, and its properties, as given in the Question, Mr. WALKER had elsewhere called attention. BRESSE's method of treating the Question is, however, quite different from the above, which is more like SCHELL's, but less diffuse.]

6405 & 6531. (By Prof. SYLVESTER, F.R.S.)—If p, q, r, s are the distances of a point in a circular cubic from the four concyclic foci A, B, C, D; prove (1) that

$$\frac{(p-q)(q-r)(r-p)}{\mathrm{ABC}} = \frac{(q-r)(r-s)(s-q)}{\mathrm{BCD}} = \frac{(r-s)(s-p)(p-r)}{\mathrm{CDA}} = \&\text{c.};$$

also prove (2) that only one proper circular cubic can be drawn having four concyclic foci at the angles of a trapezoid.

I. *Solution by* W. R. WESTROPP ROBERTS, M.A.

Let O be a centre of the quadrangle formed by A, B, C, D, the four concyclic foci of the cubic, and let the distances of O from the latter points be a, b, c, d respectively; then we have

$$\left\|\begin{matrix} p & q & r & s \\ a & b & c & d \\ 1 & 1 & 1 & 1 \end{matrix}\right\| = 0, \quad \therefore \frac{(p-q)(q-r)(r-p)}{(a-b)(b-c)(c-a)} = \frac{(s-p)(p-q)(q-s)}{(d-a)(a-b)(b-d)} = \&\text{c}.$$

Now, it is easy to see that $\Sigma'_1 p^2(b-c)(c-d)(d-a) = 0$; but since p^2, q^2, r^2, s^2 are point circles, we have also $\Sigma p^2\mathrm{BCD} = 0$, hence $(d-a)(b-c)(c-d)$ is proportional to BCD; therefore, &c.

II. *Solution by* W. J. C. SHARP, M.A.

The equation to any circular cubic can be reduced to the form

$$x(x^2+y^2) = ax^2 + 2bxy + 2fy + 2gx,$$

equivalent to $\quad (x+\mathrm{A})\left[(x-h)^2+(y-k)^2\right] = (lx+my+n)^2,$

if (h, k) be a focus, and $lx+my+n = 0$ the corresponding directrix. The equations of identity are $a = l^2 - \mathrm{A} + 2h$, $b = lm + k$, $0 = m^2 - \mathrm{A}$, $g = ln + \mathrm{A}h - \frac{1}{2}(h^2+k^2)$, $f = mn + \mathrm{A}k$ and $0 = n^2 - \mathrm{A}(h^2+k^2)$, which lead to

$$\mathrm{A}^2(\mathrm{A}^2 + a\mathrm{A} - 2g) = (f - b\mathrm{A})^2, \quad \mathrm{A}^2h^2 + 2\mathrm{A}fk - f^2 = 0,$$

$$k^2 - 2bk + 2\mathrm{A}h + b^2 - \mathrm{A}(a+\mathrm{A}) = 0,$$

so that four foci lie on each of the circles

$$\mathrm{A}^2(h^2+k^2) + 2(f-\mathrm{A}b)\mathrm{A}k + 2\mathrm{A}^3h - 2g\mathrm{A}^2 + 2bf\mathrm{A} - 2f^2 = 0,$$

and concyclic foci correspond to one value of A. Consequently

$$p : q : r : s = lx+my+n : l_1x+my+n_1 : \&\text{c.},$$

and $p-q \propto (l-l_1)x + (n-n_1)$, or as $(k_1-k)\left(\dfrac{x}{\mathrm{A}}+1\right)$, [consequently $(k_2-k_1)p + (k-k_2)q + (k_1-k)v = 0$, &c., the well known relations of the focal distances,] and

$$(p-q)(q-r)(r-p) \propto (k_1-k)(k_2-k_1)(k-k_2)\left(\frac{x}{\mathrm{A}}+1\right)^3 \propto \mathrm{ABC}\left(\frac{x}{\mathrm{A}}+1\right)^3,$$

since (h, k), (h_1, k_1), (h_2, k_2) lie on the second of the above parabolas,

therefore $\quad \dfrac{(p-q)(q-r)(r-p)}{\Delta\mathrm{ABC}} = \dfrac{(q-r)(r-s)(s-q)}{\Delta\mathrm{BCD}} = \&\text{c}.$

If two of the connectors of the four concyclic foci be parallel, the equation $\lambda(A^2h^2+2Afk-f^2)+[k^2-2bk+2Ah+b^2-A(a+A)]=0$, involves $\lambda=0$ and $A=0$, and reduces to $(k-b)^2=0$, and A, C and B, D become two pairs of coincident points.

With the fuller form $x(x^2+y^2)=ax^2+2bxy+2fy+2gx+c$ for the circular cubic, the quartic in A and the equations to the parabolas and the circle become $A^4+aA^3-2gA^2+cA=(f-bA)^2$;

$$A^2h^2+2Afk-f^2+Ac=0,\quad k^2-2bk+2Ah+b^2-A(a+A)=0;$$

$$A^2(h^2+k^2)+2Ak(f-bA)+2A^3h-2gA^2+2bfA-2f^2+2Ac=0 \qquad (1).$$

Now any bicircular quartic may be inverted into a circular cubic of this form, and if $\xi=\dfrac{h}{h^2+k^2}$, $\eta=\dfrac{k}{h^2+k^2}$, the equivalent equation

$$(x+A)[(x-h)^2+(y-k)^2]=(lx+my+n)^2$$

becomes $$\frac{x+A(x^2+y^2)}{\xi^2+\eta^2}\left\{(x-\xi)^2+(y-\eta)^2\right\}=\left\{lx+my+n(x^2+y^2)\right\}^2,$$

so that (ξ,η) is a focus, and the foci lie by fours on the circles inverse to (1), which are four in number, one for each value of A. For concyclic focal distances p, q, r, s,

$$\frac{p}{(\xi^2+\eta^2)^{\frac12}}:\frac{q}{(\xi_1^2+\eta_1^2)^{\frac12}}=lx+my+n(x^2+y^2):l_1x+m_1y+n_1(x^2+y^2)$$

$$\frac{p}{(\xi^2+\eta^2)^{\frac12}}-\frac{q}{(\xi_1^2+\eta_1^2)^{\frac12}}:\frac{q}{(\xi_1^2+\eta_1^2)^{\frac12}}-\frac{r}{(\xi_2^2+\eta_2^2)^{\frac12}}:\text{\&c.}$$

$$=\frac{\eta_1}{(\xi_1^2+\eta_1^2)^{\frac12}}-\frac{\eta}{(\xi^2+\eta^2)^{\frac12}}:\frac{\eta_2}{(\xi_2^2+\eta_2^2)^{\frac12}}-\frac{\eta_1}{(\xi_1^2+\eta_1^2)^{\frac12}}:\text{\&c.};$$

therefore $Ap+Bq+Cr=0$, &c., as is well known.

[Prof. SYLVESTER remarks that the two circular cubics belonging to a concyclic system of points A, B, C, D, are the loci of the two systems of foci of conics passing through the points; and when AC and BD become two pairs of coincident points, the second cubic becomes a circle and a straight right line. See the solutions of Quest. 1900, given on various pages of Vol. VI. of the *Reprints*, especially Art. 5 of Prof. CAYLEY's solution of the Question, where it is shown that in analytical sense, the degenerate circular cubic is a pair of coincident lines and the line at infinity.]

III. *Solution by* Professor NASH, M.A.

6405. If A, −B, C, −D denote the areas of the four triangles BCD, CDA, DAB, ABC (ABCD being in order on the circle), the given relations are equivalent to the three following equations:—

$$A+B+C+D=0,\quad Ap+Bq+Cr+Ds=0 \qquad (1, 2),$$

$$Ap^2+Bq^2+Cr^2+Ds^2=0 \qquad (3).$$

Equation (1) is evident; and (3) is satisfied if p, q, r, s be the distances from A, B, C, D of any point in the plane. For, if (x_1y_1) (x_2y_2), &c. be the coordinates of these points, and r the radius of the circle, the centre being origin, we have

$$Ap^2+Bq^2+Cr^2+Ds^2=(x^2+y^2+r^2)(A+B+C+D)$$
$$-2x(x_1A+x_2B+x_3C+x_4D)-2y(y_1A+y_2B+y_3C+y_4D) \qquad (4);$$

and since $A = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$ with similar expressions for B, C, D, the coefficient of $2x$ in (4) is $\begin{vmatrix} x_1 & x_1 & y_1 & 1 \\ x_2 & x_2 & y_2 & 1 \\ x_3 & x_3 & y_3 & 1 \\ x_4 & x_4 & y_4 & 1 \end{vmatrix} = 0,$

Similarly the coefficient of y vanishes; hence, therefore, (3) is satisfied.

It remains to show that (2) is satisfied for any point on either of the two circular cubics having A, B, C, D as foci.

By means of (1) and (3), the equation (2) may be written

$$(Ap+Bq+Cr)^2 = (A+B+C)(Ap^2+Bq^2+Cr^2), \text{ or}$$

$$A(B+C)p^2+B(C+A)q^2+C(A+B)r^2-2BCqr-2CArp-2ABpq=0 \ldots (5).$$

The discriminant of this equation vanishes, and hence the equation is equivalent to two equations of the form $lp+mq+nr=0$. The two equations are $A(B+C)p-[AB \pm (ABCD)^{\frac{1}{2}}]\,q-[AC \mp (ABCD)^{\frac{1}{2}}]\,r=0.$

Since the sum of the coefficients $=0$, these equations represent two circular cubics having A, B, C as foci. Since we might have eliminated one of the other focal distances instead of s, D of course is also a focus. Hence (2) is satisfied for both of the circular cubics having A, B, C, D as foci. Therefore, &c.

6531. Suppose AD parallel to BC, then $A+D=0$, $B+C=0$, and (5) breaks up into the two factors

$$q-r=0, \quad \text{and} \quad 2Ap-(A-B)\,q-(A+B)\,r=0.$$

The first gives the straight line bisecting AD and BC, the second gives a circular cubic having this line at axis, the collinear foci being the anti-points of the pairs AD and BC.

In order to find the true meaning of the solution $q-r=0$ we must take a different definition of a circular cubic.

If we draw the two parabolas p, p' through the four concyclic foci A, B, C, D, the circular cubics are the envelopes of the two systems of circles having their centres in p, p' and cutting the circle orthogonally. Now, if AD be parallel to BC, one of the parabolas degenerates into the pair of parallel lines AD and BC, and the envelope becomes the two pairs of anti-points of AD and BC, and the circular points at infinity.

6150. (By H. W. Harris, B.A.) — In the *Quarterly Journal of Mathematics*, Vol. I., Prof. Cayley has discussed the locus of the vertex of a triangle circumscribing a given conic and whose vertices move on given curves. In the case of the curves being both conics, the locus is of the eighth degree. Show that, in the case of all three curves being parabolas inscribed in the same triangle, the locus will reduce to a conic; and show how this last is related to the other three.

Solution by the Proposer.

It is easily proved (see Salmon's *Conics*, 5th ed., p. 359) that the locus of the *free* vertex of a triangle which circumscribes a conic, and two of whose

vertices move on confocal conics, is another conic confocal with the rest. Projecting this theorem, we have a corresponding one for conics inscribed in the same quadrilateral, and projecting one side of this quadrilateral we have the corresponding theorem for parabolas inscribed in the same triangle, and of course the conic which is the locus of the free vertex is another parabola inscribed in the same parabola. Also, since the property above of confocal conics is true for any polygon, the corresponding property of the parabolas is also true generally; that is to say, if a polygon of n sides be such that it circumscribes, while $n-1$ of its vertices describe each a parabola belonging to a system having three tangents common, then the locus of the free vertex is a similar parabola.

It may be noted that a similar proposition is true where the system of parabolas are replaced by a system of conics having double contact with each other along the same line. This is evident by projecting from the case of concentric circles, which may either be regarded as evident or as a particular case of the theorem regarding confocal conics, when the eccentricity vanishes.

6292. (By J. McDowell, M.A.)—If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, prove that

$$p_1 - \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 - \dots + \frac{1}{n}(-1)^{n-1}p_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n}.$$

6323. (By the Rev. D. Thomas, M.A.)—If n_m denote the coefficient of x^m in the expansion of $(1+x)^n$, where n is a positive integer, and if

$$F_r(n) = \frac{n_r}{1} - \frac{n_{r+1}}{2} + \dots; \text{ prove that } F_r(n) = n_{r-1}\left\{\frac{1}{r} + \dots + \frac{1}{n}\right\}.$$

6324. (By J. J. Walker, M.A.)—If m, n be positive integers, or which m is not less than n, and p_r, q_r stand for the coefficients of x^r in the development of $(1+x)^n$, $(1-x)^{-m+n-1}$ respectively, prove that

$$\frac{p_1}{q_1} - \frac{p_2}{2q_2} + \frac{p_3}{3q_3} \dots (-1)^{n-1}\frac{p_n}{nq_n} = \frac{1}{m-n+1} + \frac{1}{m-n+2} + \dots + \frac{1}{n}.$$

Solution by J. J. Walker, M.A.

Mr. McDowell's equality was given as Quest 2928 by Mr. Artemas Martin (*Reprint*, Vol. XII., pp. 80—83), and very fully discussed by Mr. Blissard, while Mr. Laverty, Prof. Wolstenholme, and others contributed an inductive solution. Before I had (quite accidentally) met with the earlier proposal of the question, it interested me, as suggesting a second form of a familiar successive differential coefficient; viz., if $u = x^m \log_e x$, then, by Leibnitz's Theorem ($n \not> m$), we have

$$\frac{d^n u}{dx^n} = x^{m-n}\left\{m \dots (m-n+1)\log_e x + p_1 m \dots (m-n+2) - p_2 m \dots (m-n+3) \right.$$
$$\left. + 2p_3 m \dots (m-n+4) + 2 \cdot 3 p_4 m \dots (m-n+5) - \dots\right\};$$

or $$\frac{1}{m\ldots(m-n+1)}\frac{d^n u}{dx^n} = x^{m-n}\left\{\log_e x + \frac{p_1}{q_1} - \tfrac{1}{2}\frac{p_2}{q_2} + \tfrac{1}{3}\frac{p_3}{q_3}\ldots(-1)^{n-1}\frac{1}{n}\frac{p_n}{q_n}\right..$$

But, by another method of differentiating (see Quest. 6552),

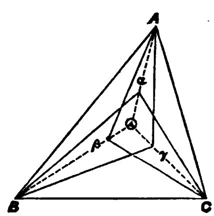
$$\frac{1}{m\ldots(m-n+1)}\frac{d^n u}{dx^n} = x^{m-n}\left(\log_e x + \frac{1}{m-n+1} + \frac{1}{m-n+2} + \ldots + \frac{1}{m}\right).$$

Thus the equality in Quest. 6324 is established, and, by making $m = n$, that given in Quests. 2928 and 6222 follows.

Writing $m-n+1 = r$ or $n = m-r+1$, it is easily seen that the equality of 6324 is transformed into what that of 6323 would become after division by n_{r-1}, and substitution of m for n.

6389. (By J. W. Russell, M.A.)—A conic cuts the sides of a triangle ABC in the pairs of points $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ respectively: if Bb_2, Cc_2 intersect in α_1; Bb_1, Cc_2 in α_2, and so on; and if $\beta_1\beta_2\beta_3\beta_4$, $\gamma_1\gamma_2\gamma_3\gamma_4$ be similarly constructed; show that the straight lines obtained by putting in various suffixes in $A\alpha$, $B\beta$, $C\gamma$ meet three by three in eight points.

Solution by W. H. Blythe, B.A.; W. B. Grove, B.A.; *and others.*



First, to prove that, if a conic cut the sides of a triangle in $a_1 a_2$, $b_1 b_2$, $c_1 c_2$, then the six straight lines Aa_1, Aa_2, Bb_1, Bb_2, Cc_1, Cc_2 touch the same conic.

We have, by Carnot's theorem,

$$Ac_1 Ac_2 \,.\, Ba_1 Ba_2 \,.\, Cb_1 Cb_2 = Ab_1 Ab_2 \,.\, Ca_1 Ca_2 \,.\, Bc_1 Bc_2.$$

Hence, by Trigonometry,

$$\sin ACc_1 \,.\, \sin ACc_2 \,.\, \&c. = \sin ABb_1 \,.\, \sin ABb_2, \&c.$$

Hence, by Salmon's *Conics*, Art. 313, the proposition follows.

Hence $A\beta C\alpha B\gamma$ is a Brianchon's hexagon; hence $A\alpha$, $B\beta$, $C\gamma$ meet in a point O.

Now α can be chosen in four ways, β then in two ways, and γ then in one way. Hence there are 8 points such as O.

6102. (By the Editor).—If a coin be thrown at random on a horizontal plane; show that (1) the probability of its reaching the plane at an inclination between α and β degrees is $\cos\alpha - \cos\beta$; and (2) the average of all the inclinations with which the coin may reach the plane is the angle subtended by an arc equal to the radius.

Solution by Prof. SEITZ, M.A.; *the* Rev. T. R. TERRY, M.A.; *and others.*

1. The number of ways the coin can reach the plane with the inclination θ is $2\pi \sin\theta$; hence the probability that the inclination will be between α and β degrees is

$$\int_\alpha^\beta 2\pi \sin\theta\, d\theta \div \int_0^{\frac{1}{2}\pi} 2\pi \sin\theta\, d\theta = \int_\alpha^\beta \sin\theta\, d\theta = \cos\alpha - \cos\beta.$$

2. The average of all the inclinations is

$$\int_0^{\frac{1}{2}\pi} \theta \,.\, 2\pi \sin\theta\, d\theta \div \int_0^{\frac{1}{2}\pi} 2\pi \sin\theta\, d\theta = \int_0^{\frac{1}{2}\pi} \theta \sin\theta\, d\theta = 1,$$

the length of the arc which measures the inclination, $= 57° \, 17' \, 44''\cdot 8$.

[The above average shows that, if the orbits of comets be supposed to be equally distributed through space, their mean inclination to the plane of the ecliptic is the angle subtended by an arc equal to the radius; a theorem whereon LIAGRE (*Calcul des Probabilités,* Art. 113) has the following note:—"Si l'on calcule, pour les comètes connues jusqu'aujourd'hui, la moyenne de l'inclinaison de leurs orbites sur l'écliptique, on trouve une valeur d'environ 50°. On doit donc conclure de ce fait que les comètes ont une tendance à *se rapprocher* du plan de l'écliptique, conclusion contraire à celle qu'a tirée Laplace" (*Théorie analytique des Probabilités,* p. 259.)

The given Question (6102) is the same in substance as the following Question (567) of the *Nouvelles Annales de Mathématiques*:—"Quelle au la probabilité que l'angle aigu formé par deux grands cercles tracés tes hasard sur une sphère soit comprise entre m degrés et n degrés?" Of this Question there is a solution in the *Annales* for August, 1861, where the probability is found to be $\cos m - \cos n$; but it is remarked that "Laplace trouve ($\cos m - \cos n : 90°$), ce qui semble inexact."]

5878 (By Prof. WOLSTENHOLME, M.A.)—The four real foci of an axial circular cubic are A, B, C, D, and the foot of the asymptote is O: O is the centroid of A, B, C, D, and if $\alpha, \beta, \gamma, \delta$ be the algebraical distances, OA, OB, OC, OD, the vector equation of the cubic (to A, B, C) is $(\beta^2-\gamma^2)\,\mathrm{AP} + (\gamma^2-\alpha^2)\,\mathrm{BP} + (\alpha^2-\beta^2)\,\mathrm{CP} = 0$. Also, if the vertices of the cubic be U, V, W, a Cartesian whose foci are U, V, W will have its vertices at A, B, C, D, and its centre (or triple focus) at O. In a Cassinoid in which, at any point P, $\mathrm{AP}\,.\,\mathrm{BP} = m\,.\,\mathrm{OA}^2$, if A′, B′ be the other two axial foci, $\mathrm{A'P}\,.\,\mathrm{B'P} = m\,.\,\mathrm{OP}^2$, O being the centre.

Solution by Prof. NASH, M.A.; Prof. EVANS, M.A.; *and others.*

The equation of an axial circular cubic, the asymptote being the axis of y, must be of the form $\quad x(x^2+y^2)+ax^2+bx+c=0$ ....(1).

Putting $y = (-1)^{\frac{1}{2}}(x-z)$ and finding the condition for equal roots, the four axial foci are determined by

$$z^4 - 2bz^2 - 8cz + b^2 - 4ac = 0 \text{ ....(2).}$$

Since this equation has no term z^3, $\Sigma z = \alpha+\beta+\gamma+\delta = 0$, and O is the

centroid of A, B, C, D. If the vector equation referred to ABC be $l \,.\, \mathrm{AP} + m \,.\, \mathrm{BP} + n \,.\, \mathrm{CP} = 0$, equation (1) must be identical with

$$l\,[(x-a)^2+y^2]^{\frac{1}{2}} + m\,[(x-\beta)^2+y^2]^{\frac{1}{2}} + n\,[(x-\gamma)^2+y^2]^{\frac{1}{2}} = 0,$$

or $$l^4\,(r^2-2ax+a^2)^2 + m^4\,(r^2-2\beta x+\beta^2)^2 + n^4\,(r^2-2\gamma x+\gamma^2)^2$$
$$= 2m^2n^2\,(r^2-2\beta x+\beta^2)^2\,(r^2-2\gamma x+\gamma^2)+\ldots \qquad \ldots\ldots(3),$$

where $r^2 = x^2+y^2$.

The terms r^4 and r^3 do not appear in (1), therefore, from (3 and 4),

$$\Sigma l^4 = 2\Sigma m^2n^2, \quad \Sigma l^4a^2 = \Sigma m^2n^2\,(\beta^2+\gamma^2), \quad l \pm m \pm n = 0 \;\ldots\ldots\ldots\ldots(4, 5, 6).$$

If we suppose AP, BP, CP to contain their signs implicitly, we may take the form (6) as $l+m+n=0$; and (5) may be written in the form

$$l^2a^2\,(m^2+n^2-l^2) + m^2\beta^2\,(n^2+l^2-m^2) + n^2\gamma^2\,(l^2+m^2-n^2) = 0.$$

Substituting $-(m+n)$ for l, we get $\dfrac{m}{\gamma^2-a^2} = \dfrac{n}{a^2-\beta^2} = \dfrac{l}{\beta^2-\gamma^2}$, therefore vector equation is $(\beta^2-\gamma^2)\,\mathrm{AP} + (\gamma^2-a^2)\,\mathrm{BP} + (a^2-\beta^2)\,\mathrm{CP} = 0 \;\ldots\ldots\ldots(7)$.

Again, if the distances of the foci of a Cartesian from the triple focus be the roots of the equation $\quad z^3-pz^2+qz-r=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$, the equation of the Cartesian is $(x^2+y^2-q)^2+4r\,(2x-p)=0$, the triple focus being the origin; and if the origin be at a distance k from the triple focus, the equation is $[(x+k)^2+y^2-q]^2+4r\,(2x+2k-p)=0$. The vertices of this Cartesian are given by

$$(x+k)^4-2q\,(x+k)^2+8r\,(x+k)+q^2-4pr = 0 \;\ldots\ldots\ldots\ldots(9).$$

If these vertices are the foci of the cubic, (2) and (9) are identical, therefore $k=0$, $b=q$, $c=-r$, $a=-p$. Therefore O the origin is the triple focus, and the single foci are given by the equation (8), $x^3+ax^2+bx+c=0$. But, from (1), this is the equation determining the vertices U, V, W of the cubic.

This result might also be deduced from the vector equations, since from the system $l \,.\, \mathrm{AU} + m \,.\, \mathrm{BU} + n \,.\, \mathrm{CU} = 0$, &c., we can obtain another system of the form $\lambda \,.\, \mathrm{AU} + \mu \,.\, \mathrm{AV} + \nu \,.\, \mathrm{AW} = 0$. For the Cassinian, if $\mathrm{OA} = a$,

$$[(x-a)^2+y^2]\,[(x+a)^2+y^2] = m^2a^4, \text{ or } (r^2+a^2)^2-4a^2x^2 = m^2a^4 \ldots\ldots\ldots(a).$$

The other foci A′, B′ are at a distance from $\mathrm{O} = a\,(1-m^2)^{\frac{1}{2}}$. Therefore

$$\mathrm{A'P}^2 \,.\, \mathrm{B'P}^2 = [r^2+a^2\,(1-m^2)]^2-4a^2x^2\,(1-m^2)$$
$$= (r^2+a^2-a^2m^2)^2-(1-m^2)\,[(r^2+a^2)^2-m^2a^4], \text{ by } (a),$$
$$= m^2\,(r^2+a^2)^2-2m^2a^2\,(r^2+a^2)+m^2a^4 = m^2r^4.$$

Hence we have $\qquad \mathrm{A'P} \,.\, \mathrm{B'P} = m \,.\, \mathrm{OP}^2.$

4753. (By Prof. Crofton, F.R.S.)—If 1, 2, 3, 4 are four conclyclic points on a Cassinian oval whose foci are F, F′, show that a bicircular oval can be drawn with 1, 2, 3, 4 as foci to pass through F, F′. Show also that the tangents at F, F′ are *double tangents;* each touching the bicircular oval in a second point, and each passing through the centre of the circle 1234.

Solution by Prof. NASH, M.A.; Prof. SCHEFFER, M.A.; *and others.*

The equation of a Cassinian is $(x^2+y^2+c^2)^2-4c^2x^2=m^4$ .... (1), the foci of which are $x=\pm c$, $y=0$. Let the equation of the circle through 1, 2, 3, 4 be $x^2+y^2=2ax+2\beta y+\gamma$ .... (2), a, β being the coordinates of the centre O. Substituting for x^2+y^2 in (1), the equation of a conic through 1, 2, 3, 4 is

$$(2ax+2\beta y+\gamma+c^2)^2-4c^2x^2=m^4 \quad\text{.....................} (3).$$

The asymptotes of this conic are parallel to the lines $(ax+\beta y)^2-c^2x^2=0$, and therefore perpendicular to the lines OF, OF′, *i.e.*,

$$(\beta x-ay)^2-c^2(y-\beta)^2=0.$$

Hence OF, OF′ are double tangents to a bicircular quartic, having (2) for a circle of inversion, and (3) for the corresponding focal conic. [See Prof. CASEY's paper on *Bicircular Quartics*, Art. 47.] It remains to show that F, F′ are points of contact. If p be the perpendicular from O upon the asymptote, and r the distance from O of a point of contact of the double tangent OF; then, since the two points of contact are inverse with respect to (2), and are also equidistant from the asymptote $2pr=r^2+a^2+\beta^2+\gamma$. Now, taking the asymptote $2ax+2\beta y+\gamma+c^2-2cx=0$,

$$p=\frac{2a^2+2\beta^2+\gamma+c^2-2ca}{2\left[(a-c)^2+\beta^2\right]^{\frac12}}=\frac{\mathrm{OF}^2+a^2+\beta^2+\gamma}{2\mathrm{OF}};$$

hence F is one point of contact of the double tangent. The latter part may be otherwise proved as follows:—The sixteen points of contact of the eight double tangents of a bicircular quartic lie by fours upon four concentric circles; the common centre being the centre of the focal conics, and each circle cuts orthogonally one of the circles of inversion. Hence the points of contact of the double tangents OF and OF′ lie upon the circle

$$x^2+\left(y+\frac{c^2+\gamma}{2\beta}\right)^2=\left(\frac{c^2+\gamma}{2\beta}\right)^2+c^2+\gamma-\gamma,\ \text{or}\ x^2+y^2+\frac{c^2+\gamma}{\beta}y=c^2,$$

and this circle evidently passes through F, F′.

6554. (By C. E. BICKMORE, M.A.)—Solve the equations

$$xyz+x-y-z=a(1+yz-zx-xy),\quad xyz+y-z-x=b(1+zx-xy-yz),$$
$$xyz+z-x-y=c(1+xy-yz-zx).$$

Solution by Rev. J. L. KITCHIN, M.A.; J. O'REGAN; *and others.*

The equations are equivalent to

$$\frac{1-y\,.\,1-z}{1+y\,.\,1+z}=\frac{1+a\,.\,1-x}{1-a\,.\,1+x},\ \text{\&c. \&c.};$$

hence
$$\frac{1-x}{1+x}\left(\frac{1+a}{1-a}\right)^{\frac12}=\frac{1-y}{1+y}\left(\frac{1+b}{1-b}\right)^{\frac12}=\left(\frac{1-z}{1+z}\right)\left(\frac{1+c}{1-c}\right)^{\frac12}$$
$$=\left(\frac{1+a}{1-a}\cdot\frac{1+b}{1-b}\cdot\frac{1+c}{1-c}\right)^{\frac12}.$$

5083 & 5180. (By Prof. Wolstenholme, M.A.)—A bicircular quartic whose real foci lie on a circle is the locus of a point P, which moves so that the ratio AP . BP : CP . DP is constant, A, B, C, D being four fixed points on the focal circle; and there are for each curve two sets of such fixed points. Given the foci of the curve, show how to obtain these points; or, given one set of the points, show how to obtain the other set and the foci.

A circular cubic whose four real foci lie on a circle is the locus of a point P, which moves so that AP . BP = CP . DP, where A, B, C, D are four fixed points on the focal circle, and there are for each curve two sets of such points. Given the foci of the curve, show how to obtain these points; or, given one set of the points, show how to obtain the other set and the foci.

Solution by Prof. Nash, M.A; Christine Ladd; *and others.*

It will be sufficient to prove Quest. 5180, as Quest. 5083 can at once be deduced by inversion from any point.

To find the locus of a point P which moves so that AP . BP = CP . DP, A, B, C, D being four given points on a circle. Here P is evidently the locus of the intersection of the circles AP = k . CP, k . BP = DP. Now, k being variable, AP = k . CP represents a system of circles coaxal with the point circles A and C, and therefore orthotomic to the given circle, and similarly for k . BP = DP. Also, if Q, R be the centres of corresponding circles, we must have AQ : CQ = DR : BR; hence the centres Q, R divide AC and DB proportionately; therefore QR envelopes a parabola touching AC and BD.

If P, P′ be the two points of intersection of these circles, since they both cut the circle ABCD orthogonally, P, P′ must pass through its centre O, and are also inverse points with respect to this circle. Therefore P, P′ lie upon a circular cubic, having as foci the four points in which the parabola cuts the given circle. [See Prof. Casey's paper on *Bicircular Quartics* in the Royal Irish Academy's *Transactions*, Vol. 24, p. 492.]

Taking the values $k = 0$, $k = \infty$, we see that AD, BC are tangents to the parabola. If the foci are given, the problem becomes—To inscribe in a given circle a quadrilateral whose sides shall touch a given parabola; a problem which is not generally possible, except when the inverse of the focus with respect to the circle lies upon the directrix. If we take the particular case when the four foci are collinear, the equation of the cubic is $x(x^2+y^2)+ax^2+bx+c=0$; and, in order that this may be the locus of a point P satisfying the relation AP . BP = CP . DP, where A, B, C, D are points on the axis of the curve, we must have the relation

$$a^3-4ab+8c=0.$$

The other cases can be deduced from this by inversion.

5773. (By J. L. McKenzie, B.A.) — Through any point P on a circular cubic draw any circle, cutting the cubic again in A, B, C; through A, B draw any circle cutting the cubic in D, E; let PD cut the cubic in Q, QC in R, and RE in S; prove that PS is the tangent to the cubic at P.

Solution by Prof. Nash, M.A.; Prof. Matz, M.A.; *and others.*

Using the notation (AB) to denote the third point in which the line AB meets the cubic, we have [(AB), (CP)] and also [(AB), (DE)] each parallel to the asymptote; hence CP, DE meet the curve in the same point K, or more briefly (CP) = (DE) = K.

Again, D, E, Q, R being four points on the cubic, we have

$$[(DE)(QR)] = [(DQ)(RE)],$$

since the pairs of lines DE, QR and DQ, RE may be considered as two conics through the four points; therefore (KC) = (PS); but (KC) = P, therefore (PS) = P, *i.e.* the line PS meets the curve again at P, and is therefore the tangent at P.

6563. (By Prof. Sylvester, F.R.S.)—1. The velocities of three bodies in the direction of the lines joining them with their centre of gravity, and the lengths of these lines, being given; find the transverse velocities [that is, perpendicular respectively to the former ones] in the plane of the bodies, in order that the centre of gravity, and the principal axes through it of the system, may remain at rest for a moment of time.

2. Given the distances of three bodies from their centre of gravity, determine the angles which they make respectively with the principal axes of the system, and the values of the moments of inertia in respect to these axes.

Solution by J. J. Walker, M.A.

By "bodies" is to be understood masses (m, m', m'') whose linear dimensions are infinitesimal in comparison with their distances (say a, b, c) from their common centre of gravity (G). Let θ, θ', θ'' be the angles which these distances respectively make with a principal axis through G, in the plane of the three bodies; then the following relations hold:—

$$ma^2 \sin 2\theta + m'b^2 \sin 2\theta' + m''c^2 \sin 2\theta'' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$ma \sin\theta + m'b \sin\theta' + m''c \sin\theta'' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$ma \cos\theta + m'b \cos\theta + m''c \cos\theta'' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Squaring (2), (3), after transferring the last terms to the other side, and adding, we get

$$\cos(\theta-\theta') = \cos\gamma = (-m^2a^2 - m'^2b^2 + m''^2c^2) : 2mm'bc,$$

giving $\quad \sin(\theta-\theta') = \sin\gamma = (2\Sigma m'^2m''^2b^2c^2 - \Sigma m^4a^4)^{\frac{1}{2}} : 2mm'bc;$

with similar values for $\cos(\theta-\theta'') = \cos\beta$, $\sin(\theta-\theta'') = \sin\beta$. Writing $\theta+\gamma$ for θ', $\theta+\beta$ for θ'' in (1), and dividing by $\cos 2\theta$, θ or $\frac{1}{2}\pi+\theta$ is determined by

$$\tan 2\theta = -(+m'b^2 \sin 2\gamma + m''c^2 \sin 2\beta) : (ma^2 + m'b^2 \cos 2\gamma + m''c^2 \cos 2\beta),$$

and thus the second part of the question is solved. For the first part, let δa, $a\delta\theta$ ... be the displacements along, and transversely to, a ... in the element of time δt; then G and the principal axes will plainly be unmoved if equations (1), (2), (3) hold after substitution of $a+\delta a$, $\theta+\delta\theta$ ... for a, θ .... This condition leads to

$$ma^2 \cos 2\theta \,.\, \delta\theta + m'b^2 \cos 2\theta' \,.\, \delta\theta' + m''c^2 \cos 2\theta'' \,.\, \delta\theta'' = -a\delta a \sin 2\theta - \ldots \quad \text{..(i.)}$$
$$ma \cos \theta \,.\, \delta\theta + m'b \cos \theta' \,.\, \delta\theta' + m''c \cos \theta'' \,.\, \delta\theta'' = -\delta a \sin \theta - \ldots \quad \text{...(ii.)}$$
$$ma \sin \theta \,.\, \delta\theta + m'b \sin \theta' \,.\, \delta\theta' + m''c \sin \theta'' \,.\, \delta\theta'' = \delta a \cos \theta + \ldots \quad \text{..(iii.)},$$
equations which determine $a\delta\theta,\ b\delta\theta',\ c\delta\theta'' : \delta t$, in terms of
$$v = \delta a : \delta t,\ \ v' = \delta a' : \delta t,\ \ v'' = \delta a'' : \delta t.$$

[Prof. SYLVESTER remarks that the above questions have an important application in the method employed by him to throw into equations the problem of the motion of three bodies acted on by their mutual attractions, which he reduces to a system of three differential equations of the second order, and one of the first, between the time and four quantities, viz., the distances from the centre of gravity and the angular velocity of the principal axis, equivalent (leaving out t) to six linear simultaneous equations between seven quantities (p, q, r ; p', q', r' ; ω).

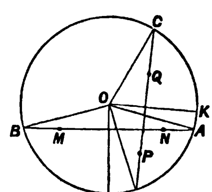
JACOBI, in the "Elimination of the Nodes," reduces the problem to depend on five equations of the first order and one of the second, of course equivalent (leaving out t) to six linear simultaneous equations between seven variables. JACOBI transforms the problem, at starting, into one relating only to two bodies; but Prof. SYLVESTER adheres to the original data throughout, thus adopting a method which he believes to be more natural, and leading to a result superior in point of form.]

5533. (By the EDITOR.)—Through two points taken at random anywhere inside a circle a chord is drawn, and a second chord through two other such random points: show that (1) the probability that these two chords will intersect inside the circle is $\frac{1}{3} + \frac{245}{72\pi^2}$; also (2), if through a third pair of such random points, a third chord be drawn, show that the respective probabilities that the three chords will intersect inside the circle in 0, 1, 2, 3 points are

$$p_0 = \frac{1}{3} - \frac{245}{36\pi^2} + \frac{23023}{576\pi^4},\quad p_1 = \frac{2}{5} + \frac{155}{36\pi^2} - \frac{55055}{864\pi^4},$$
$$p_2 = \frac{1}{5} + \frac{115}{72\pi^2} + \frac{13013}{1728\pi^4},\quad p_3 = \frac{1}{15} + \frac{65}{72\pi^2} + \frac{7007}{432\pi^4}.$$

Solution by Professor SEITZ, M.A.

1. Let M, N and P, Q be two pairs of random points within the circle whose centre is O, and AB, CD the chords drawn through them. Draw the radii OH, OK perpendicular to AB, CD. Let OA = r, AM = u, MN = v, CP = w, PQ = x, AB = u', CD = w', $\angle$ AOH = θ, $\angle$ COK = ϕ, $\angle$ HOK = μ, and ω = the angle OH makes with a fixed radius. Then $u' = 2r \sin\theta$, $w' = 2r \sin\phi$; an element of the circle at M is $r \sin\theta\, d\theta\, du$, at N it is $d\omega\, v\, dv$, at P it is $r \sin\phi\, d\phi\, dw$, and at Q it is $d\mu\, x\, dx$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ,



0 and θ, and θ and $\frac{1}{2}\pi$; of μ, $\theta-\phi$ and $\theta+\phi$ when $\phi < \theta$, and $\phi-\theta$ and

$\phi+\theta$ when $\phi > \theta$, and the result doubled; of ω, 0 and 2π; of u, 0 and u'; of v, 0 and u, and doubled; of w, 0 and w'; and of x, 0 and w, and doubled, Hence, since the whole number of ways the four points can be taken is $\pi^4 r^8$, the required probability is

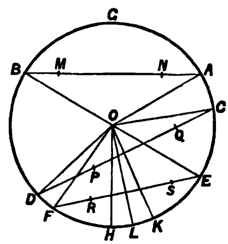
$$p = \frac{8}{\pi^4 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{u'}\int_0^{u}\left\{\int_0^{w'}\int_0^{w}\left[\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi} r\sin\phi\, d\phi\, d\mu\right.\right.$$

$$\left.\left.+\int_{\theta}^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta} r\sin\phi\, d\phi\, d\mu\right] dw\, x\, dx\right\} r\sin\theta\, d\theta\, d\omega\, du\, v\, dv$$

$$= \frac{64}{3\pi^3}\int_0^{\frac{1}{2}\pi}\left\{\tfrac{4}{3}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\sin^4\phi\, d\phi\, d\mu + \tfrac{4}{3}\int_{\theta}^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta}\sin^4\phi\, d\phi\, d\mu\right\}\sin^4\theta\, d\theta$$

$$= \frac{512}{9\pi^3}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}\phi\sin^4\phi\, d\phi + \int_0^{\frac{1}{2}\pi}\theta\sin^4\phi\, d\phi\right\}\sin^4\theta\, d\theta$$

$$= \frac{32}{9\pi^3}\int_0^{\frac{1}{2}\pi}(3\pi\theta - 3\theta^2 + 3\sin^2\theta + \sin^4\theta)\sin^4\theta\, d\theta = \frac{1}{3} + \frac{245}{72\pi^2}.$$



2. Let R, S be the third pair of random points, and EF the chord drawn through them. Draw the radius OL perpendicular to EF. Let p_0, p_1, p_2, p_3 be the respective probabilities that AB, CD, EF will intersect in 0, 1, 2, 3 points. Let $\mathrm{ER}=y$, $\mathrm{RS}=z$, $\mathrm{EF}=y'$, $\angle\mathrm{EOL}=\psi$, and $\angle\mathrm{KOL}=\rho$. Then $y'=2r\sin\psi$; an element of the circle at R is $r\sin\psi\, d\psi\, dy$, and at S it is $d\rho\, z\, dz$.

For non-intersection, the chord CD being between AB and EF, the limits of θ are 0 and π; of ϕ, 0 and θ; of ψ, 0 and ϕ; of μ, $\phi-\theta$ and $\theta-\phi$; and of ρ, $\psi-\phi$ and $\phi-\psi$. The result of these integrations must be multiplied by 6 to allow for the interchange of the chords.

For non-intersection, the chord EF having its ends in the arc BD, the limits of θ are 0 and π; of ϕ, 0 and θ; of ψ, 0 and $\theta-\phi$; of μ, $2\psi-\theta+\phi=\alpha$ and $\theta-\phi$; and of ρ, $\phi+\psi$ and $\theta+\mu-\psi=\beta$. This result must be multiplied by 4 to allow for the positions of EF, in which its ends are in the arc AC, and for the positions of CD and EF, in which their ends are in the arc AGB. The limits of ω are 0 and π; of u, 0 and u'; of v, 0 and u, and doubled; of w, 0 and w'; of x, 0 and w, and doubled; of y, 0 and y'; and of z, 0 and y, and doubled. Hence, since the whole number of ways the six points can be taken is $\pi^6 r^{12}$, we have

$$p_0 = \frac{48}{\pi^6 r^{12}}\int_0^{\pi}\int_0^{\theta}\int_0^{\phi}\int_{\phi-\theta}^{\theta-\phi}\int_{\psi-\phi}^{\phi-\psi}\int_0^{\pi}\int_0^{u'}\int_0^{u}\int_0^{w'}\int_0^{w}\int_0^{y'}\int_0^{y} r^3\sin\theta\sin\phi\sin\psi$$

$$\times\, d\theta\, d\phi\, d\psi\, d\mu\, d\rho\, d\omega\, du\, v\, dv\, dw\, x\, dx\, dy\, z\, dz$$

$$+ \frac{32}{\pi^6 r^{12}}\int_0^{\pi}\int_0^{\theta}\int_0^{\theta-\phi}\int_{\alpha}^{\theta-\phi}\int_{\phi+\psi}^{\beta}\int_0^{\pi}\int^{u'}\int_0^{u}\int_0^{w'}\int_0^{w}\int_0^{y'}\int_0^{y} r^3\sin\theta\sin\phi\sin\psi$$

$$\times\, d\theta\, d\phi\, d\psi\, d\mu\, d\rho\, d\omega\, du\, v\, dv\, dw\, x\, dx\, dy\, z\, dz$$

$$= \frac{1024}{27\pi^5}\int_0^{\pi}\int_0^{\theta}\left\{\int_0^{\phi}\int_{\phi-\theta}^{\theta-\phi}\int_{\psi-\phi}^{\phi-\psi} 3\sin^4\psi\, d\psi\, d\mu\, d\rho \right.$$

$$\left. + \int_0^{\theta-\phi}\int_0^{\theta-\phi}\int_{\phi+\psi}^{\theta} 2\sin^4\psi\, d\psi\, d\mu\, d\rho \right\} \sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{4096}{27\pi^5}\int_0^{\pi}\int_0^{\theta}\left\{\int_0^{\phi} 3(\theta-\phi)(\phi-\psi)\sin^4\psi\, d\psi \right.$$

$$\left. + \int_0^{\theta-\phi}(\theta-\phi-\psi)^2\sin^4\psi\, d\psi \right\}\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{16}{27\pi^5}\int_0^{\pi}\int_0^{\theta}\left\{32(\theta-\phi)^3 + 144\phi^2(\theta-\phi) - 60(\theta-\phi) - 144(\theta-\phi)\sin^2\phi \right.$$

$$\left. -48(\theta-\phi)\sin^4\phi + 32\sin 2(\theta-\phi) - \sin 4(\theta-\phi)\right\}\sin^4\theta\sin^4\phi\, d\theta\, d\phi$$

$$= \frac{1}{81\pi^5}\int_0^{\pi}(360\theta^4 - 1665\theta^2 - 1296\theta^2\sin^2\theta - 432\theta^2\sin^4\theta - 4020\theta\sin\theta\cos\theta$$

$$-408\theta\sin^3\theta\cos\theta + 5685\sin^2\theta + 559\sin^4\theta$$

$$+248\sin^6\theta + 36\sin^8\theta)\sin^4\theta\, d\theta$$

$$= \frac{1}{3} - \frac{245}{36\pi^2} + \frac{23023}{576\pi^4}.$$

We will next find the probability that the chords CD, EF will both intersect AB. By reference to the first part of the solution, we readily see that the probability that a third chord will also intersect AB is

$$P = \frac{256}{3\pi^5}\int_0^{\frac{1}{2}\pi}\left\{\tfrac{4}{3}\int_0^{\theta}\int_{\theta-\phi}^{\theta+\phi}\sin^4\phi\, d\phi\, d\mu + \tfrac{4}{3}\int_{\theta}^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta}\sin^4\phi\, d\phi\, d\mu\right\}^2\sin^4\theta\, d\theta$$

$$= \frac{16384}{27\pi^5}\int_0^{\frac{1}{2}\pi}\left\{\int_0^{\theta}\phi\sin^4\phi\, d\phi + \int_{\theta}^{\frac{1}{2}\pi}\theta\sin^4\phi\, d\phi\right\}^2\sin^4\theta\, d\theta$$

$$= \frac{64}{27\pi^5}\int_0^{\frac{1}{2}\pi}(3\pi\theta - 3\theta^2 + 3\sin^2\theta + \sin^4\theta)^2\sin^4\theta\, d\theta = \frac{2}{15} + \frac{155}{108\pi^2} + \frac{97097}{5184\pi^4}.$$

Now P is evidently equal to the probability that CD, EF will both intersect AB, and not each other, together with p_3; hence we have

$$\tfrac{1}{3}p_2 + p_3 = P \text{..................................} (1).$$

The probability, p, that AB and CD will intersect, is equal to the probability that they will intersect each other, and not EF, plus the probability that CD, EF will both intersect AB, and not each other, plus the probability that AB, EF will both intersect CD, and not each other, plus p_3; therefore $\frac{1}{3}p_1 + \frac{2}{3}p_2 + p_3 = p$, also $p_1 + p_2 + p_3 = 1 - p_0$ ......(2, 3). From (1), (2), (3), knowing the values of P, p, p_0, we find for p_1, p_2, p_3 the values given in the question.

5454. (By Prof. Townsend, F.R.S.)—A solid ellipsoid of uniform density being supposed to revolve round its least axis of figure, and to

carry with it a surrounding envelope of homogeneous incompressible fluid of different density, the entire mass attracting according to the ordinary law of the inverse square of the distance; required the several conditions requisite to the permanent assumption of the ellipsoidal form by the free surface of the fluid.

Solution by the PROPOSER.

The entire force acting on any superficial element of the fluid consisting of three parts, arising respectively from—(1) the attraction of the fluid, and of a hypothetical mass, of equal density, occupying the space of the solid; (2) the attraction or repulsion, as the case may be, of a second hypothetical mass, having the difference of densities of the solid and fluid, and occupying again the space of the former; and (3) the rotation of the entire mass, as a whole, round its axis of figure; and the components, parallel to the axes, of the first as well as of the third parts, being, for the assumed form of free surface, proportional to the corresponding coordinates of the element at every point; in order that those of the second part also should be proportional to them, as they must necessarily be for the permanence of the assumed form, the surface of the fluid, on the known principles of the theory of ellipsoidal attraction for the law of the inverse square of the distance, must be confocal with that of the solid; which, accordingly, is the first condition requisite to the possibility of the assumed form.

This condition being supposed fulfilled, and the three semi-axes a_1, b_1, c_1 of the outer being thus completely determined from the three a, b, c of the inner ellipsoid by virtue of the known volume of the fluid; it is further necessary that the external ellipsoid, thus completely determined should be of the Jacobian form,—that, viz., (see BESANT's *Hydromechanics*, 3rd ed., Art. 114,) for which $\displaystyle\int_0^1 \frac{u^2(1-u^2)(1-\lambda^2\mu^2u^2)\,du}{[(1+\lambda^2u^2)(1+\mu^2u^2)]^{\frac{3}{2}}} = 0$;

whence $a_1^2 = c_1^2(1+\lambda^2)$ and $b_1^2 = c_1^2(1+\mu^2)$; which, accordingly, is the second condition requisite to the possibility of the assumed form.

This condition also being supposed fulfilled, it is further requisite, finally, that the velocity of rotation ω have the definite value given (see reference already made) by the equation

$$\omega^2 = \frac{3M}{c_1^2}\int_0^1 \frac{u^2(1-u^2)\,du}{[(1+\lambda^2u^2)(1+\mu^2u^2)]^{\frac{3}{2}}},$$

when M = the entire mass of the solid and fluid taken together; which, accordingly, is the third condition requisite to the possibility of the assumed form.

6565. (By Prof. MINCHIN, M.A.)—Prove that the areas of roulettes, in their most general forms, follow the law of circular transformation obtained by STEINER, for the areas of pedals (see WILLIAMSON's *Integral Calculus*, 3rd ed., p. 202); and thus, in virtue of a general kinematical theorem, deduce at once KEMPE's theorem (*ibid.* p. 210).

Solution by C. LEUDESDORF, M.A.

In the *Messenger* for May, 1878, I gave a proof of this, which I repeat here, as it was not worked out in full, being precisely similar to the proof there given of a theorem relating to areas of pedals.

The most general motion of a point in a plane being reducible to a case of roulettes, let a curve roll upon a fixed curve, carrying points A, B, C, D, which describe, in consequence, areas (A), (B), (C), (D). Let D, referred to ABC, have areal coordinates x, y, z, and let P be the instantaneous point of contact of the two curves. As in WILLIAMSON's *Integral Calculus*, p. 205,

$$(A) = (S) + \tfrac{1}{2}\int\left(1+\frac{\rho}{\rho'}\right)PA^2 d\omega,$$

with similar equations for B, C, D. Therefore

$$(D)-x(A)-y(B)-z(C) = \tfrac{1}{2}\int\left(1+\frac{\rho}{\rho'}\right)(PD^2-x\,PA^2-y\,PB^2-z\,PC^2)\,d\omega$$

$$= -\tfrac{1}{2}\int\left(1+\frac{\rho}{\rho'}\right)(a^2yz+b^2zx+c^2xy)\,d\omega$$

(by an easily proved geometrical relation)

$$= k\,(a^2yz+b^2zx+c^2xy),\ k \text{ being a constant.}$$

If (D) is made constant, the locus of D is therefore, in general, a circle; which gives KEMPE's theorem, as originally deduced by him in the *Messenger* for April, 1878. If, in addition, (A) = (B) = (C), the locus of D is the circle round ABC, as in WILLIAMSON's *Integral Calculus*, p. 210.

[Prof. MINCHIN states that when, two months ago, he proved and developed this theorem, he looked at the papers referred to without finding any treatment of the results of HOLDITCH, AMSLER, and others, from the Roulette point of view; and he adds that, while the theorems in the *Messenger* are limited to *closed* paths, the above theorem is not so limited.]

6533. (By Prof. CROFTON, F.R.S.)—Prove that

$$(r+1)^r = r^r + 2^0\,.\,r(r-1)^{r-1} + 3^1\,.\,\frac{r(r-1)}{1\,.\,2}(r-2)^{r-2} + 4^2\,.\,\frac{r(r-1)(r-2)}{1\,.\,2\,.\,3}$$
$$\times(r-3)^{r-3} + 5^3\,.\,\frac{r(r-1)(r-2)(r-3)}{1\,.\,2\,.\,3\,.\,4}(r-4)^{r-4} + \ldots + (r+1)^{r-1}.$$

Solution by J. J. WALKER, M.A.

In the Mathematical Society's *Proceedings*, Vol. xi., p. 108, I have given the development

$$D^n u\phi^{n+1} = \phi D^n u\phi^n + n\phi'\phi D^{n-1}u\phi^{n-1} + n_2 D\phi'\phi^2\,.\,D^{n-2}u\phi^{n-2}\ldots$$
$$\ldots + nD^{n-2}\phi'\phi^{n-1}\,.\,Du\phi + uD^{n-1}\phi'\phi^n,$$

u and ϕ being any functions of x, and ϕ' being $D\phi$.

If $u = e^{(r-n)x}$, $\phi = e^x$, both sides being divisible by $e^{(r+1)x}$, we obtain the more general development

$$(r+1)^n = r^n + 2^0 n(r-1)^{n-1} + 3^1 n_2(r-2)^{n-2} + 4^2 n_3(r-3)^{n-3} + \ldots + (n+1)^{n-1};$$

whereof Prof. CROFTON's is the particular case in which $n = r$ or $u = 1$.

6545. (By ELIZABETH BLACKWOOD.)—In the integral

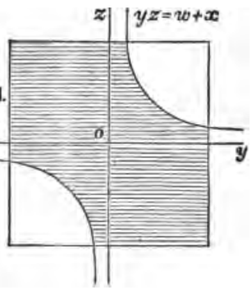
$$\int dw \int dx \int dy \int dz\, \phi\,(w,\, x,\, y,\, z)$$

the four variables are each restricted between the limits a and $-\imath$, and the integration is further restricted by the condition $w+x > yz$; find the limits of integration when z varies first, y next, and w last.

I. *Solution by* G. F. WALKER, M.A.; Prof. MATZ, M.A.; *and others.*

Take $w+x$ constant and trace the curve $yz = w+x$; we must take Fig. 1, if $w+x$ be positive, and Fig. 2 if it be negative, the square being $x = \pm a$, $y = \pm b$ in each case, and the point being restricted to the shaded area, and Fig. 3 giving the limits of integration of $w+x$. Let P denote $\phi\,(x,\, y,\, z,\, w)$. The integral with respect to $y+z$ is

FIG. 1.



$$\int_{-a}^{a}\int_{-a}^{a} dy\,dz\,\mathrm{P} - \int_{\frac{w+x}{a}}^{a}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P} - \int_{-a}^{-\frac{w+x}{a}}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P}$$

FIG. 2.

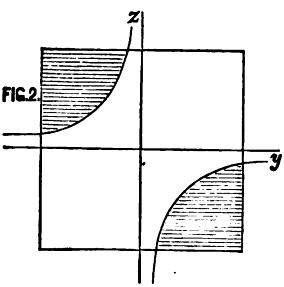
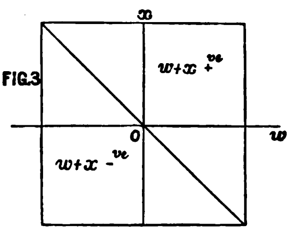


FIG. 3



if $w+x$ be positive, and $\displaystyle\int_{-\frac{w+x}{a}}^{a}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P} + \int_{-a}^{\frac{w+x}{a}}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P}$

if $w+x$ be negative, and the complete result is

$$\int_{-a}^{a}\int_{-w}^{a}\left[\int_{-a}^{a}\int_{-a}^{a} dy\,dz\,\mathrm{P} - \int_{\frac{w+x}{a}}^{a}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P} - \int_{-a}^{-\frac{w+x}{a}}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P}\right] dw\,dx$$

$$+\int_{-a}^{a}\int_{-a}^{-w}\left[\int_{-\frac{w+x}{a}}^{a}\int_{-a}^{\frac{w+x}{y}} dy\,dz\,\mathrm{P} + \int_{-a}^{\frac{w+x}{a}}\int_{\frac{w+x}{y}}^{a} dy\,dz\,\mathrm{P}\right] dw\,dx.$$

If $a > 2$, this solution is correct. If $a < 2$, we must add

$$\int_{a^2-a}^{a}\int_{a^2-w}^{a}\int_{-a}^{a}\int_{-a}^{a} \mathrm{P}\,dw\,dx\,dy\,dz.$$

II. *Solution by* W. B. GROVE, B.A.; H. McCOLL, B.A.; *and others.*

Let $A \equiv w_{1'.2}\, x_{1'.2}\, y_{1'.2}\, z_{1'.2}\, p\,(w+x-yz)$, where the symbols have the meanings attached to them in Mr. McCOLL's first Paper in *Proceedings of the London Mathematical Society*. Then, according as $A > < 2$, it will be found that

$$A = w_{1'.2}\left\{x_{1'.3}\left(y_{3'.4}\, z_{1'.2}+y_{4'.2}\, z_{1'.3}+y_{1'.3}\, z_{3'.2}\right)\right.$$
$$\left.+x_{3'.2}\left(y_{3'.2}\, z_{1'.3}+y_{1'.4}\, z_{3'.2}\right)\right\},$$

or
$$A = \left(w_{1'.3}\, x_{4'.3}+w_{3'.2}\, x_{1'.3}\right)\left(y_{3'.4}\, z_{1'.2}+y_{4'.2}\, z_{1'.3}+y_{1'.3}\, z_{3'.2}\right)$$
$$+\left(w_{1'.4}\, x_{3'.2}+w_{4'.2}\, x_{3'.5}\right)\left(y_{3'.2}\, z_{1'.3}+y_{1'.4}\, z_{3'.2}\right)$$
$$+w_{1'.3}\, x_{1'.4}\, y_{1'.2}\, z_{1'.2}.$$

The limits of integration, which in these statements are denoted by suffixes, may be ascertained by referring to the subjoined table:—

$a_1 = 2$	$w_1 = a$	$x_1 = a$	$y_1 = a$	$z_1 = a$
	$w_2 = -a$	$x_2 = -a$	$y_2 = -a$	$z_2 = -a$
	$w_3 = a^2 - a$	$x_3 = -w$	$y_3 = \dfrac{w+x}{a}$	$z_3 = \dfrac{w+x}{y}$
	$w_4 = a - a^2$	$x_4 = a^2 - w$		
		$x_5 = -a^2 - w$	$y_4 = -\dfrac{w+x}{a}$	

6548. (By W. J. C. SHARP, M.A.)—If the normals at four points on an ellipse meet in a point, prove that the sum of the excentric angles at these points is an odd multiple of π.

Solution by Prof. SCOTT, M.A.; Prof. MATZ, M.A.; *and others.*

Three normals meet in a point if $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha) = 0$, and the fourth goes through intersection of first and second if

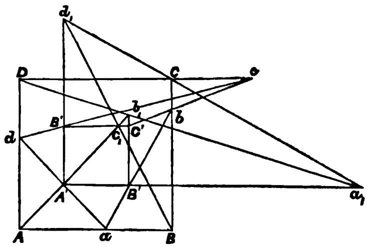
$$\sin(\alpha+\delta)+\sin(\beta+\delta)+\sin(\alpha+\beta) = 0\,;$$

hence, subtracting, $4\cos\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\gamma-\delta)\cos\frac{1}{2}(\alpha+\beta+\gamma+\delta) = 0$, or $\alpha+\beta+\gamma+\delta = (2m+1)\,\pi$.

5309. (By Professor WOLSTENHOLME, M.A.)—The two concentric squares ABCD, A′B′C′D′ have their sides parallel, and the side of the larger equal to the sum of side and diagonal of the smaller; prove that (1) an infinite number of sets of four points a, b, c, d can be taken on the

sides of either, such that the lines joining them in order, ab, bc, cd, da, pass through the corners of the other [thus, a, b, c, d lie on AB, BC, CD, DA, and ab, bc, cd, da pass through B′, C′, D′, A′; also a_1, b_1, c_1, d_1 lie on A′B′, B′C′, C′D′, D′A′, and a_1b_1, b_1c_1, c_1d_1, d_1a_1 pass through D, A, B, C]; and (2) $abcd$ and $a_1b_1c_1d_1$ are each a mean proportional between the two.

Solution by the PROPOSER.



Area of $abcd$ = square A′B′C′D′ + 4 equal triangles $\left(\text{each} = \text{area of square} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\right)$ = square A′B′C′D′ $(1-\sqrt{2})$ = mean proportional between the two squares.

So, area $a_1b_1c_1d_1$ = 4 equal Δ^s $\left(\text{each} = \text{square ABCD} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\right)$ − square ABCD = square ABCD $(\sqrt{2}-1)$ = square A′B′C′D′ $(\sqrt{2}+1)$ = mean proportional between the two.

The value $\frac{17}{12}$ has been taken in the figure for $\sqrt{2}$, and is sufficiently near. The corresponding system of poristic equations is

$$\frac{1}{x_2}+\frac{1}{1-x_1}=\frac{1}{x_3}+\frac{1}{1-x_2}=\frac{1}{x_4}+\frac{1}{1-x_3}=\frac{1}{x_1}+\frac{1}{1-x_4}=\frac{1}{k},$$

where $k=\frac{1}{2}$, or $(2\pm\sqrt{2})^{-1}$; but the first value reduces the second square to a point, and the in- and circum-scribed figure to a single straight line.

6540. (By Prof. MATZ, M.A.)—A, B, C, D play a set of games in which two are partners against the other two, any two being equally likely to be partners, and drawn games being impossible. If A's chance of winning a single game with B, C, D for partner be p, q, r respectively; find the chances that A, B, C, D respectively will win a out of $a+b$ games.

Solution by W. B. GROVE, B.A.; ELIZABETH BLACKWOOD; *and others.*

Let p_A denote the chance that A wins a single game, $q_A = 1-p_A$, and so on; then $p_A = \frac{1}{3}(p+q+r)$, $p_B = \frac{1}{3}[p+(1-q)+(1-r)]$,

$p_C = \frac{1}{3}[(1-p)+q+(1-r)]$, $p_D = \frac{1}{3}[(1-p)+(1-q)+r]$;

and, by a well-known formula, the chance that A wins a games out of $a+b$, with similar expressions for B, C, D, is

$$p_A^a\left\{1+aq_A+\frac{a(a+1)}{1.2}q_A^2+\ldots+\frac{a(a+1)\ldots(a+b-1)}{b!}q_A^b\right\}.$$

5642. (By E. B. ELLIOTT, M.A.)—If $\frac{x-A}{l} = \frac{y-B}{m} = \frac{z-C}{n}$ be the type of a given doubly-infinite set of straight lines, l, m, n being direction-cosines, and A, B, C given functions of these which make $A\,dl + B\,dm + C\,dn$ the exact differential of a function $\phi(l, m, n)$; prove that the polar tangential equation of a class of surfaces cutting all the straight lines orthogonally is $p = \phi(l, m, n) + c$, c being any constant.

Solution by W. J. C. SHARP, B.A.; Prof. JOHNSON, M.A.; *and others.*

Now $\frac{x-A}{l} = \frac{y-B}{m} = \frac{z-C}{n}$ is the equation to one of the lines, and therefore $lx + my + nz = p$ is the equation to the tangent plane to the orthogonal surface, and $(l+\delta l)\,x + (m+\delta m)\,y + (n+\delta n)\,z = p + \delta p$ that to an adjacent tangent plane; hence, at the intersection

$$x\delta l + y\delta m + z\delta n = \delta p,$$

therefore $\quad A\delta l + B\delta m + C\delta n + k\,(l\delta l + m\delta m + n\delta n) = \delta p,$

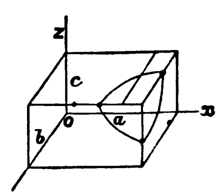
or $\quad d\,.\,\phi(lmn) + \tfrac{1}{2}kd\,(l^2 + m^2 + n^2) = \delta p;$

but $l^2 + m^2 + n^2 = 1$, therefore $d\,(l^2 + m^2 + n^2) = 0$, therefore $\phi(lmn) + c = p$.

6523. (By G. F. WALKER, M.A.)—Three quantities lie between $(0, a)$, $(0, b)$, $(0, c)$ respectively; find the chance that their product is less than d^3 $(d^3 < abc)$. [Suggested by Miss BLACKWOOD's Question 6414.]

I. *Solution by* Prof. PRATT, M.A.; *the* PROPOSER; *and others.*

The required chance is the volume of the rectangle abc on the origin side of the surface $xyz = d^3$,



$$\text{and chance} = \frac{abc - \int_{\frac{d^3}{bc}}^{a} \int_{\frac{d^3}{cx}}^{b} y\left(c - \frac{d^3}{xy}\right) dx\,dy}{abc},$$

$$= \frac{d^3}{abc}\left\{1 + \log\frac{abc}{d^3} + \tfrac{1}{2}\left[\log\frac{abc}{d^3}\right]^2\right\}.$$

II. *Solution by* W. B. GROVE, B.A.; H. McCOLL, B.A.; *and others.*

The chance required $= AQ\iiint dx\,dy\,dz \div A\iiint dx\,dy\,dz$; where x, y, z are the three quantities; A denotes the statement that they lie between $(0, a)$ $(0, b)$ $(0, c)$ respectively; and Q the statement that their product is

less than d^3; therefore

$$A \equiv x_{1'.0}\, y_{1'.0}\, z_{1'.0},$$

$$AQ \equiv x_{1'.0}\, y_{1'.0}\, z_{1'.2'.0}$$

$$= x_{2'.0}\, y_{1'.0}\, z_{1'.0} + x_{1'.2}\, y_{2'.0}\, z_{1'.0} + x_{1'.0}\, y_{1'.2}\, z_{2'.0};$$

$x_1 = a$	$y_1 = a$	$z_1 = c$
$x_2 = \dfrac{d^3}{bc}$	$y_2 = \dfrac{d^3}{cx}$	$z_2 = \dfrac{d^3}{xy}$

hence the chance required is

$$\frac{\displaystyle\int_0^{\frac{d^3}{bc}}\int_0^b\int_0^c dx\,dy\,dz + \int_{\frac{d^3}{bc}}^a\int_0^{\frac{d^3}{xy}}\int_0^c dx\,dy\,dz + \int_0^a\int_{\frac{d^3}{cx}}^b\int_0^{\frac{d^3}{cx}} dx\,dy\,dz}{\displaystyle\int_0^a\int_0^b\int_0^c dx\,dy\,dz}$$

$$= \frac{d^3}{abc}\left\{1 + \log\frac{abc}{d^3} + \tfrac{1}{2}\left(\log\frac{abc}{d^3}\right)^2\right\}.$$

This expression can be extended in a similar way to the case of more than three quantities. If there are n quantities lying between $(0, a)$ $(0, b)$ $(0, c)$... respectively, and d^n be $< abc...$, the chance that their product is less than d^n is

$$\frac{d^n}{abc...}\left\{1 + \log\frac{abc...}{d^n} + \frac{1}{2!}\left(\log\frac{abc...}{d^n}\right)^2 + \ldots\ldots + \frac{1}{(n-1)!}\left(\log\frac{abc...}{d^n}\right)^{n-1}\right\}.$$

6051. (By D. EDWARDES.)—A particle describes an ellipse under an attraction to the centre. Prove that (1) the sum of the squares of the velocities at the extremities of a pair of conjugate diameters is constant; (2) the sum of the squares of the reciprocals of the velocities at any two points the directions at which include a right angle is constant.

Solution by the Rev. T. R. TERRY, M.A.; D. WICKERSHAM; *and others.*

1. If CP, CD be conjugate diameters, it is known that velocity at P is parallel and proportional to CD. But $CP^2 + CD^2 = \text{constant}$.

2. If CD, Cd are at right angles, $\dfrac{1}{CD^2} + \dfrac{1}{Cd^2} = \text{constant}$.

6541. (By Professor NASH, M.A.)—A cardioid being generated by a circle rolling upon a fixed equal circle; prove that the polar reciprocal of the cardioid with respect to the fixed circle is a nodal circular cubic; that there are two real axial foci besides the double focus at the node; and that the vector equation is $3\rho_1 \pm 2\rho_2 - \rho_3 = 0$, where ρ_2 is the distance from the double focus.

I. *Solution by* Prof. WOLSTENHOLME, M.A.; CHRISTINE LADD; *and others.*

The equation of the tangent of the cardioid (origin at centre of fixed circle) is $x \sin 3\theta - y \cos 3\theta = 3a \sin\theta$; hence, for the corresponding point on the reciprocal polar, $\dfrac{x}{\sin 3\theta} = \dfrac{-y}{\cos 3\theta} = \dfrac{a}{3 \sin\theta}$,

whence $x = a\left(1 - \frac{4}{3}\sin^2\theta\right)$, $x^2 + y^2 = \dfrac{a^2}{9\sin^2\theta}$, and the equation is

$$(x^2 + y^2)(a - x) = \tfrac{4}{27}a^3.$$

If $x + iy = p$ be tangent from a focus, the equation

$$p(x - iy)\left(\frac{a}{p}(x + iy) - x\right) = \frac{4}{27}\,\frac{a^3}{p^2}(x + iy)^2$$

must have equal roots; hence

$$4\left(a - p - \frac{4}{27}\frac{a^3}{p^2}\right)\left(-a - \frac{4}{27}\frac{a^3}{p^2}\right) = \left(p - \frac{8}{27}\frac{a^3}{p^2}\right)^2,$$

or $\quad (p - 2a)^2 - \dfrac{16}{27}\dfrac{a^3}{p^2}(p + a - a + p) = 0$; or $p^2(p - 2a)^2 = \dfrac{32a^3}{27}p$,

which has roots 0, $\frac{2}{3}a$, $\frac{2}{3}a$, $\frac{8}{3}a$. (Of course the double focus is the node, and is known beforehand as the point reciprocal to the double tangent of the cardioid $x = \frac{3}{2}a$.)

If r_1, r_2, r_3 be the three vectors for any point on the curve

$$r_1^2 = \frac{4}{27}\frac{a^3}{a - x}, \quad r_2^2 = \left(x - \tfrac{2}{3}a\right)^2 + y^2 = \frac{4}{27}\frac{a^3}{a - x} - \tfrac{4}{3}ax + \tfrac{4}{9}a^2 = \tfrac{4}{3}a\,\frac{\left(x - \frac{2}{3}a\right)^2}{a - x},$$

$$r_3^2 = \frac{4}{27}\frac{a^3}{a - x} - \tfrac{16}{3}ax + \tfrac{64}{9}a^3 = \frac{4}{27}\,\frac{a(6x - 7a)^2}{a - x};$$

hence, for any point on the oval ($x < \frac{2}{3}a$), we have

$$\frac{r_1}{a} = \frac{r_2}{2a - 3x} = \frac{r_3}{7a - 6x} \quad \text{or} \quad 3r_1 + 2r_2 = r_3.$$

For the sinuous branch, $\dfrac{r_1}{a} = \dfrac{r_2}{3x - 2a} = \dfrac{r_3}{7a - 6x}$, and $3r_1 = 2r_2 + r_3$.

II. *Solution by* Professor TOWNSEND, M.A., F.R.S.

The cardioid being a tricuspal quartic, symmetrical with respect to an axis of figure, touching the axis cuspally at its single real point of meeting with the quiescent circle of its ordinary generation, intersecting it again orthogonally at three times the same distance in the opposite direction from the centre of the circle, passing cuspally through the two circular points at infinity, and having real double contact with a bitangent chord perpendicular to the axis, at points determining an equilateral triangle with the centre of the circle, of which its axial cusp is the centre of figure; its polar reciprocal to the quiescent circle is consequently a crunodal cubic, symmetrical with respect to the same axis of figure, intersecting the axis doubly at angles of 60° at a distance from the centre towards its axial cusp equal to two-thirds of the radius of the circle, intersecting it again singly at right angles at half the same distance in the opposite

direction from the centre, passing inflexionally through the two circular points at infinity, and having real inflexional contact with a terminal asymptote perpendicular to the axis, the tangent, viz., to the circle at its axial cusp.

Hence, denoting by a the radius of the circle, and taking for axes of x and of y respectively the axis and asymptote of the cubic, the positive direction of x being towards the centre of the circle; the equation

$$y^2x + (x-\tfrac{1}{3}a)^2(x-\tfrac{4}{3}a) = 0,$$

or its equivalent $\qquad (x^2+y^2)\,x-2ax^2+a^2x-\frac{4}{27} = 0,$

is easily seen to satisfy all the aforesaid particulars of the latter, and shows that, as stated in the question, the curve has three real axial foci situated at the distances $+a$, $+\frac{1}{3}a$, and $-\frac{5}{3}a$ from the origin, the first coinciding with the centre of the circle and the second with the node of the curve; a circle of evanescent radius round each as centre having, as may be easily seen, double contact with the curve, real in the case of the nodal and imaginary in those of the other two foci.

To find the vectorial equation of the curve referred to its three foci, or, which is the same thing, to determine the values of the three ratios $\lambda:\mu:\nu$ in the linear equation $\lambda\rho+\mu\sigma+\nu\tau = 0$, where ρ, σ, τ are the three focal vectors, in the above order of the foci, of any point on the curve. Since, for the nodal point on the axis $\lambda \pm 0\mu \cdot 3\nu = 0$, for the apsidal point on the axis $\lambda + 3\mu + 9\nu = 0$, and for the inflexional point on the asymptote $\lambda + \mu + \nu = 0$, therefore at once, as stated in the question, $\lambda:\mu:\nu \quad 3:2:-1$ for the loop, and $= 3:-2:-1$ for the remainder of the curve.

6550. (By W. R. Westropp Roberts, M.A.)—A circular elastic lamina, of uniform thickness and isotropic material, is capable of motion round an axis passing through its centre and perpendicular to its plane; being given the strength of the material, determine the angular velocity which is just sufficient to produce permanent alteration of form.

Solution by the Proposer.

Let u, v be the projections of the displacement of any point P, on two rectangular axes in the plane of the lamina, which must evidently be of the form $\qquad u = x\,.f(r), \quad v = y\,.f(r);$

whence, with the usual notation,

$$\theta = 2f(r)+rf'(r), \quad N_1 = \lambda\theta+2\mu\left\{f(r)+\frac{x^2}{r}.f'(r)\right\}, \quad T_3 = 2\mu\,xyf'(r);$$

and these values, substituted in the equation

$$\frac{dN_1}{dx}+\frac{dT_3}{dy}+m\Omega^2x = 0, \quad \text{give } (\lambda+2\mu)\frac{d\theta}{dr} = -m\Omega^2 r;$$

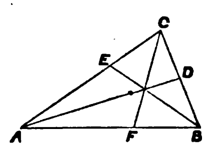
consequently $\qquad f(r) = \alpha+\frac{\beta}{r^2}-\gamma\frac{r^2}{4}, \text{ where } \gamma = \frac{m\Omega^2}{2(\lambda+2\mu)}.$

Now, we must have $\beta = 0$, since both u and v vanish when $r = 0$; hence
$$\theta = 2\alpha - \gamma r^2,\ f(r) = \alpha - \gamma\,\frac{r^2}{4}.$$

Following LAME's notation, R_1, the normal elastic force on an element plane perpendicular to any radius r, is found from the above value of N_1 by putting $x = r$; and Φ_2, the normal elastic force on an element plane of a section by a plane through the axis of rotation, by putting $x = 0$; hence
$$R_1 = 2\alpha(\lambda+\mu) - \gamma r^2(\lambda + \tfrac{3}{2}\mu), \quad \Phi_2 = 2\alpha(\lambda+\mu) - \gamma r^2(\lambda + \tfrac{1}{2}\mu).$$
If we suppose $R_1 = 0$ when $r = a$, the radius of the plate, and A the greatest value of Φ_2, which occurs when $r = 0$, we find
$$\Omega^2 = \frac{4A}{ma^2}\cdot\frac{(\lambda+2\mu)}{(2\lambda+3\mu)}.$$

4188. (By Dr. HART.)—If the angles of a plane triangle be bisected by the given lines a, b, c, terminating in the opposite sides respectively, it is required to find the sides.

Solution by Prof. EVANS, M.A.; *the* Rev. U. J. KNISELY, M.A.; *and others.*



Let ABC be a triangle similar to the required triangle, $AD = a\beta$, $BE = b\beta$, $CF = c\beta$, $AB = 1$, $BC = x$, $AC = y$. Then $AF = \dfrac{y}{x+y}$, $FB = \dfrac{x}{x+y}$, $BD = \dfrac{x}{1+y}$, $DC = \dfrac{xy}{1+y}$, $CE = \dfrac{xy}{x+1}$, $EA = \dfrac{y}{x+1}$;

and since $AC\,.\,BC = AF\,.\,FB + CF^2$,
$$AB\,.\,AC = BD\,.\,DC + AD^2,\quad AB\,.\,BC = AE\,.\,EC + BE^2,$$
we have $CF^2 : AD^2 = AC\,.\,BC - AF\,.\,FB : AB\,.\,AC - BD\,.\,DC$,
$$CF^2 : BE^2 = AC\,.\,BC - AF\,.\,FB : AB\,.\,BC - AE\,.\,EC;$$
or
$$c^2 : a^2 = xy\left\{1-\left(\frac{1}{x+y}\right)^2\right\} : y\left\{1-\left(\frac{x}{1+y}\right)^2\right\} \quad\ldots\ldots\ldots\ldots(1),$$
$$c^2 : b^2 = xy\left\{1-\left(\frac{1}{x+y}\right)^2\right\} : x\left\{1-\left(\frac{y}{x+1}\right)^2\right\} \quad\ldots\ldots\ldots\ldots(2).$$
When a, b, c are given in numbers, x and y may be found from (1) and (2); and thence the sides $\dfrac{1}{\beta}$, $\dfrac{x}{\beta}$, $\dfrac{y}{\beta}$ of the required triangle, since
$$\beta^2 = \frac{xy}{c^2}\left\{1-\left(\frac{1}{x+y}\right)^2\right\}.$$

[Dr. HART remarks that he is convinced that the sides of the triangle cannot be explicitly exhibited in terms of the three given lines.]

6281. (By Prof. WOLSTENHOLME, M.A.)—Given two real foci and the real asymptote of a circular cubic; prove that the locus of the other real foci is a parabola passing through the two given foci and having its axis along the given asymptote.

Solution by W. R. WESTROPP ROBERTS, M.A.

Let the two given foci be $(a,\ 0), (-a, 0)$, and the given asymptote $px+qy=1$; and let the vector equation of the cubic for these two foci and a third (for the infinite branch) be $lr_1+mr_2 = (l+m)\,r_3$, where

$$r_1 = \left\{(x-a)^2+y^2\right\}^{\frac{1}{2}},\quad r_2 = \left\{(x+a)^2+y^2\right\}^{\frac{1}{2}},\quad r_3 = \left\{(x-\mathrm{X})^2+(y-\mathrm{Y})^2\right\}^{\frac{1}{2}},$$

then
$$r_1 = (x^2+y^2)^{\frac{1}{2}}\left\{1-\frac{2ax-a^2}{2\,(x^2+y^2)}-\frac{4\,a^2x^2}{8\,(x^2+y^2)^2}+\dots\right\},$$

$$r_2 = (x^2+y^2)^{\frac{1}{2}}\left\{1+\frac{2ax+a^2}{2\,(x^2+y^2)}-\frac{4a^2x^2}{8\,(x^2+y^2)^2}+\dots\right\},$$

$$r_3 = (x^2+y^2)^{\frac{1}{2}}\left\{1-\frac{2\,(x\mathrm{X}+y\mathrm{Y})-\mathrm{X}^2-\mathrm{Y}^2}{2\,(x^2+y^2)}-\frac{4\,(x\mathrm{X}+y\mathrm{Y})^2}{8\,(x^2+y^2)^2}\right\};$$

and we shall have for the asymptote

$$\left(2ax-a^2+\frac{a^2q^2}{p^2+q^2}\right)-m\left(2ax+a^2-\frac{a^2q^2}{p^2+q^2}\right)$$
$$= (l+m)\left\{2x\mathrm{X}+2y\mathrm{Y}-\mathrm{X}^2-\mathrm{Y}^2+\frac{(q\mathrm{X}-p\mathrm{Y})^2}{p^2+q^2}\right\};$$

and, making this coincide with $px+qy = 1$, we get

$$(l+m)\,\frac{2\mathrm{Y}}{q} = (l+m)\left\{\mathrm{X}^2+\mathrm{Y}^2-\frac{(q\mathrm{X}-p\mathrm{Y})^2}{p^2+q^2}-a^2+\frac{a^2q^2}{p^2+q^2}\right\},$$

or
$$\frac{2\,(p^2+q^2)}{q}\,\mathrm{Y} = (p\mathrm{X}+q\mathrm{Y})^2-p^2a^2;\ \text{therefore \&c.}$$

[One significant term was omitted, in each expansion, in the Solution given of this Question on p. 115 of Vol. XXXIV. of the *Reprint.*]

5344. (By S. TEBAY, B.A.)—n grocers have $a_1, a_2, \dots a_n$ lbs. of tea respectively ($a_1 < a_2 <$ &c.); all sell a certain number of lbs. at the same price, and afterwards the remainders at a different price, and all realize equal sums from the two sales. Show how this can be done, and deduce a simple particular solution.

Solution by Prof. EVANS, M.A.; *the* PROPOSER; *and others.*

Let P be the price at which $x_1, x_2, \dots x_n$ lbs. are sold, and P′ the price at which the remainders are sold. Then we have

$$\mathrm{P}x_1+\mathrm{P}'(a_1-x_1) = \mathrm{P}x_2+\mathrm{P}'(a_2-x_2) = \mathrm{P}x_3+\mathrm{P}'(a_3-x_3) = \text{\&c.}$$

Therefore $$\frac{P}{P'} = 1 - \frac{a_2 - a_1}{x_2 - x_1} = 1 - \frac{a_3 - a_2}{x_3 - x_2} = \&c.;$$

whence $$x_1(a_3 - a_2) - x_2(a_3 - a_1) + x_3(a_2 - a_1) = 0,$$
$$x_2(a_4 - a_3) - x_3(a_4 - a_2) + x_4(a_3 - a_2) = 0, \&c.$$

There are $n-2$ equations, so that, if x_1, x_2 be assumed, the others follow.

Let a_1, a_2, &c. be in arithmetical progression, d the common difference; then $x_1 - 2x_2 + x_3 = 0$, $x_2 - 2x_3 + x_4 = 0$, &c.; so that x_1, x_2, &c. are in arithmetical progression. Let d' be the difference, then $\frac{P}{P'} = 1 - \frac{d}{d'}$. This being reduced or modified, the numerator can be taken for P, and the denominator for P′; the limiting inequality for d' being

$$x_1 + (n-1)\,d' < a_n.$$

Let $a_1 = 10$, $a_2 = 13$, $a_3 = 16$, $a_4 = 19$; so that $d = 3$; and take $x_1 = 2$. Then $2 + 3d' < 19$, or $d' < 5\frac{2}{3}$; therefore $d' = 4$ or 5. Let $\frac{P}{P'} = 1 - \frac{3}{5} = \frac{2}{5}$; and take P = 2*s*., P′ = 5*s*. The number of pounds at 2*s*. = 2, 7, 12, 17; and the number at 5*s*. = 8, 6, 4, 2; and each realizes 44 shillings.

5121. (By S. Tebay, B.A.)—If n letters be formed in groups of p letters in each, no q letters being repeated, but every q of the n letters exhibited; then, on erasing x letters, shew, x being less than q, that the number of groups containing r of the x letters is

$$N_r = \frac{x!}{r!\,(x-r)!} \cdot \frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-r)!} \cdot \frac{(n-p)!}{(n-p-x+r)!}.$$

Solution by Belle Easton; *the* Proposer; *and others.*

To find N_x, compare the combinations in q, each containing the x letters; thus, $$\frac{(p-x)!}{(q-x)!\,(p-q)!}\,N_x = \frac{(n-x)!}{(q-x)!\,(n-q)!}, \text{ and } N_x = \frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-x)!}.$$

To find N_{x-1}, compare the combinations in q, each containing $x-1$ of the x letters; thus,

$$\frac{[p-(x-1)]!}{[q-(x-1)]!\,(p-q)!}\,N_{x-1} + x\,\frac{(p-x)!}{[q-(x-1)]!\,(p-q-1)!}\,N_x$$
$$= x\,\frac{(n-x)!}{[q-(x-1)]!\,(n-q-1)!};$$

therefore $$N_{x-1} = x\,\frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-x+1)!}\,(n-p).$$

To find N_{x-2}, compare the combinations in q, each containing $x-2$ of the x letters; thus,

$$\frac{[p-(x-2)]!}{[q-(x-2)]!\,(p-q)!}\,N_{x-2}+(x-1)\,\frac{[p-(x-1)]!}{[q-(x-2)]!\,(p-q-1)!}\,N_{x-1}$$

$$+\frac{x(x-1)}{1\,.\,2}\cdot\frac{(p-x)!}{[q-(x-2)]!\,(p-q-2)!}\,N_x$$

$$=\frac{x(x-1)}{1\,.\,2}\cdot\frac{(n-x)!}{[q-(x-2)]!\,(n-q-2)!};$$

therefore $N_{x-2}=\dfrac{x(x-1)}{1\,.\,2}\cdot\dfrac{(n-x)!\,(p-q)!}{(n-q)!\,(p-x+2)!}\,(n-p)(n-p-1).$

And so on. Thus we find

$$N_{x-r}=\frac{x!}{r!\,(x-r)!}\cdot\frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-x+r)!}\cdot\frac{(n-p)!}{(n-p+r)!};$$

or, writing r for $x-r$,

$$N_r=\frac{x!}{r!\,(x-r)!}\cdot\frac{(n-x)!\,(p-q)!}{(n-q)!\,(p-r)!}\cdot\frac{(n-p)!}{(n-p-x+r)!}.$$

If u_n denote the whole number of combinations,

$$u_{n-x}=u_n-(N_1+N_2+\ldots+N_x).$$

5846. (By W. H. H. Hudson, M.A.)—If the normal at a point P of the lemniscate meet the axis in G, and if GQ, CR be drawn perpendicular to CG, CP respectively to meet the circle described on the line joining the poles of the lemniscate as diameter in Q, R; prove that

$$GQ : CQ = CQ^2 : RP^2.$$

Solution by J. A. Kealy, M.A.; J. O'Regan; *and others.*

Let the equation of lemniscate be $(x^2+y^2)^2 = 2a^2(x^2-y^2)$, where $2a$ is the distance between the poles, and let P be the point (x, y), then we have $CQ = a$, $RP^2 = a^2+CP^2 = a^2+x_1^2+y_1^2$;

Normal at P, $y-y_1=\dfrac{y_1^3+x_1^2y_1+a^2y_1}{x_1^3+x_1y_1^2-a^2x_1}(x-x')$, $\quad CG=\dfrac{2a^2x_1}{a^2+x_1^2+y_1^2}$,

$$GQ=\frac{a^3}{a^2+x_1^2+y_1^2},\quad GQ : CQ = a^2 : a^2+x_1^2+y_1^2 = CQ^2 : RP^2.$$

6595. (By the late S. M. Drach, F.R.A.S.)—

If six-digit number N be so reckoned
That twice (6th less 3rd) plus thrice (5th less 2nd),
With 4th less 1st, announce sum sevenfold,
Whole number N by *sevens* can be told.

Quadruple (6th less 3rd) plus thrice (5th less 2nd)
With 1st less 4th, is test if *thirteen*'s beckoned.

Solution by W. B. Grove, B.A.; Prof. Droz, M.A.; *and others.*

Let $$N \equiv 10^5 r_6 + 10^4 r_5 + 10^3 r_4 + 10^2 r_3 + 10 r_2 + r_1$$
$$= 7ar_6 - 2r_6 + 7br_5 - 3r_5 + 7cr_4 - r_4 + 7a'r_3 + 2r_3 + 7b'r_2 + 3r_2 + r_1$$
$$= 7M - \{2(r_6 - r_3) + 3(r_5 - r_2) + (r_4 - r_1)\}.$$
Also $$N = 13ar_6 + 4r_6 + 13br_5 + 3r_5 + 13cr_4 - r_4 + 13a'r_6 - 4r_3 + 13b'r_5 - 3r_2 + r_1$$
$$= 13M + 4(r_6 - r_3) + 3(r_5 - r_2) + (r_1 - r_4).$$
The process can be similarly extended to any number of digits.

6552. (By J. J. Walker, M.A.)—If $u = x^n \log x$, prove that there is a form of $\dfrac{d^r u}{dx^r}$ other than that given by Leibnitz's Theorem; viz.,
$$n \,.\, n-1 \ldots n-r+1 \,.\, x^{n-r}\left(\log x + \frac{1}{n-r+1} + \frac{1}{n-r+2} \ldots + \frac{1}{n}\right).$$

Solution by the Proposer.

If $u = x^n \log x$, we have $\dfrac{1}{n}\dfrac{du}{dx} = x^{n-1}\log x + x^{n-1}$,
$$\frac{1}{n\,.\,n-1}\frac{du^2}{dx^2} = x^{n-2}\log x + \frac{x^{n-2}}{n-1} + \frac{x^{n-2}}{n};$$
and generally $$\frac{1}{n \ldots (n-r+1)}\frac{d^r u}{dx^r} = x^{n-r}\log x + \frac{x^{n-r}}{n-r+1} + \frac{x^{n-r}}{n-r+2} \ldots + \frac{x^{n-r}}{n}.$$
For, differentiating again, and dividing by $n-r$,
$$\frac{1}{n \ldots (n-r)}\frac{d^{r+1}u}{dx^{r+1}} = x^{n-r-1}\log x + \frac{x^{n-r-1}}{n-r} + \frac{x^{n-r-1}}{n-r+1} + \ldots + \frac{x^{n-r-1}}{n}.$$
Thus, if the formula holds for one value of r, it holds for the next integer; but it holds for $r = 1$, $r = 2$.

4185. (Proposed by W. H. H. Hudson, M.A.)—A series of ellipsoids are constructed, each of which is the ellipsoid of gyration of the preceding one about its centre; find the form of the nth ellipsoid so constructed, the first being given, and all the ellipsoids being supposed of uniform density; prove also that the ultimate form of the nth ellipsoid, when n is increased indefinitely, is that of a sphere.

Solution by J. L. KITCHIN, M.A.; A. MARTIN, M.A.; *and others.*

Let the original ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$;

then, putting $d^2=a^2+b^2+c^2$, the successive ellipsoids of gyration will be

$$\frac{x^2}{\frac{b^2+c^2}{5}}+\frac{y^2}{\frac{a^2+c^2}{5}}+\frac{z^2}{\frac{a^2+b^2}{5}}=1,\qquad \frac{x^2}{\frac{d^2+a^2}{5^2}}+\frac{y^2}{\frac{d^2+b^2}{5^2}}+\frac{z^2}{\frac{d^2+c^2}{5^2}}=1,$$

$$\frac{x^2}{\frac{3d^2-a^2}{5^3}}+\frac{y^2}{\frac{3d^2-b^2}{5^3}}+\frac{z^2}{\frac{3d^2-c^2}{5^3}}=1,\ \text{\&c., \&c.}$$

Now, when n is very large, $\frac{a^2}{5^n}$, &c. are very small; hence

$$\frac{x^2}{\frac{md^2}{5^n}}+\text{\&c.}\ldots\ldots=1,\quad\text{therefore}\quad x^2+y^2+z^2=\frac{md^2}{5^n},\ \text{a sphere.}$$

5194. (By A. W. PANTON, M.A.)—If the coordinates (x, y, z) of a point on a unicursal cubic be proportional to $a_1\theta^3+b_1\theta^2+c_1\theta+d_1$, $a_2\theta^3+b_2\theta^2+c_2\theta+d_2$, $a_3\theta^3+b_3\theta^2+c_3\theta+d_3$; show that the three values of θ at the points of inflexion are the roots of the cubic equation

$$\begin{vmatrix} 1 & -3\theta & 3\theta^2 & -\theta^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}=0.$$

Solution by Prof. NASH, M.A.; Prof. MATZ, M.A.; *and others.*

Any straight line $\lambda x+\mu y+\nu z=0$ meets the cubic in three points given by the equation

$$(\lambda a_1+\mu a_2+\nu a_3)\,\theta^3+(\lambda b_1+\mu b_2+\nu b_3)\,\theta^2+(\lambda c_1+\mu c_2+\nu c_3)\,\theta + (\lambda d_1+\mu d_2+\nu d_3)=0\ldots\ldots\ldots(1),$$

and if this line be the tangent at the point of inflexion $\theta=\alpha$, this equation must reduce to $k(\theta-\alpha)^3=0$; whence α is given by the determinant equation obtained by eliminating λ, μ, ν, k from the equations to the left; therefore

$$\left|\begin{matrix} \lambda a_1+\mu a_2+\nu a_3- k=0 \\ \lambda b_1+\mu b_2+\nu b_3+3k\alpha=0 \\ \lambda c_1+\mu c_2+\nu c_3+3k\alpha^2=0 \\ \lambda d_1+\mu d_2+\nu d_3+k\alpha^3=0 \end{matrix}\right|,\qquad \begin{vmatrix} 1 & -3\alpha & 3\alpha^2 & -\alpha^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}=0.$$

Using Prof. WOLSTENHOLME's notation, the equation is

$$D\alpha^3+3C\alpha^2+3B\alpha+A=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If $\lambda x+\mu y+\nu z=0$ be the axis of inflexion, the equations (1) and (2) must have the same roots. Therefore

$$\frac{\lambda a_1+\mu a_2+\nu a_3}{D}=\frac{\lambda b_1+\mu b_2+\nu b_3}{3C}=\frac{\lambda c_1+\mu c_2+\nu c_3}{3B}=\frac{\lambda d_1+\mu d_2+\nu d_3}{A}.$$

These three equations are not independent, but, solving the first two, we obtain $\dfrac{\lambda}{(b_2c_3)+3(d_2a_3)} = \dfrac{\mu}{(b_3c_1)+3(d_3a_1)} = \dfrac{\nu}{(b_1c_2)+3(d_1a_2)}$.

For a unicursal quartic given by the equations

$$x = a_1\theta^4+b_1\theta^3+c_1\theta^2+d_1\theta+e_1,\quad y = a_2\theta^4+b_2\theta^3+c_2\theta^2+d_2\theta+e_2,$$
$$z = a_3\theta^4+b_3\theta^3+c_3\theta^2+d_3\theta+e_3;$$

the equation giving the values of θ at the points of inflexion is

$$\begin{vmatrix} 6\theta^2a_1+3\theta b_1+c_1, & 3\theta^2b_1+4\theta c_1+3d_1, & \theta^2c_1+3\theta d_1+6e_1 \\ 6\theta^2a_2+3\theta b_2+c_2, & 3\theta^2b_2+4\theta c_2+3d_2, & \theta^2c_2+3\theta d_2+6e_2 \\ 6\theta^2a_3+3\theta b_3+c_3, & 3\theta^2b_3+4\theta c_3+3d_3, & \theta^2c_3+3\theta d_3+6e_3 \end{vmatrix} = 0,$$

or
$$2\theta^6(a_1b_2c_3)+6\theta^5(a_1b_2d_3)+\theta^4\left\{12(a_1b_2e_3)+6(a_1c_2d_3)\right\}$$
$$+\theta^3\left\{16(a_1c_2e_3)+3(b_1c_2d_3)\right\}+\theta^2\left\{6(b_1c_2e_3)+12(a_1d_2e_3)\right\}$$
$$+6\theta(b_1d_2e_3)+2(c_1d_2e_3)=0;$$

where $(a_1b_2c_3)$ denotes the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

6586. (By D. Edwardes.)—A string of length $2l$ is suspended from two points in the same horizontal line distant $2a$ from each other. If the distance between the points be slightly increased by the quantity 2δ, prove that the vertex of the catenary will ascend through the space $\dfrac{z^2+cz-l^2}{cl-az-ac}\cdot\delta$, where c is tension at lowest point, and z the depth of lowest point below the line of suspension.

Solution by the Rev. T. R. Terry, M.A.; Lizzie A. Kittudge; *and others.*

With the given notation, $l = \frac{1}{2}c\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)$, $l^2+c^2 = (z+c)^2$......(1, 2), where l remains constant, but z and c vary with a.

From (1), $\quad 0 = \frac{1}{2}c\left(e^{\frac{a}{c}}-e^{-\frac{a}{c}}\right)\dfrac{dc}{da} - \frac{1}{2}a\left(e^{\frac{a}{c}}+e^{-\frac{a}{c}}\right)\dfrac{dc}{da}+\frac{1}{2}c\left(e^{\frac{a}{c}}+e^{-\frac{a}{c}}\right),$

or $\quad \dfrac{dc}{da} = \dfrac{-(z+c)\,c}{cl-az-ac}.$

From (2), $\quad c = (z+c)\left(\dfrac{dz}{dc}+1\right)$ or $\dfrac{dz}{dc} = -\dfrac{z}{z+c}$;

therefore $\dfrac{dz}{da} = \dfrac{zc}{cl-az-ac} = -\dfrac{z^2+zc-l^2}{cl-az-ac}$, which is the required result.

[The string is supposed uniform, and c is the length of the string whose weight is equal to the tension at the lowest point.]

6058. (By F. MORLEY, B.A.)—Two equal perfectly elastic particles fall simultaneously from points in the same vertical line at heights h, h' from a fixed perfectly elastic plane. Show that the height of the point of their first collision from the plane is, x being integral,

$$\frac{[(2x+1)^2h'-h][h-(2x-1)^2h']}{16x^2h'}, \text{ where } h>(2x-1)^2h', \quad <(2x+1)^2h'.$$

Solution by the Rev. T. R. TERRY, M.A.; J. O'REGAN; *and others.*

Suppose the lower ball strike the plane x times before collision. Time elapsed before this x^{th} impact $= (2x-1)\left(\frac{2h'}{g}\right)^{\frac{1}{2}}$. At this instant velocity of upper ball $= (2x-1)(2gh')^{\frac{1}{2}}$, and distance travelled $= (2x-1)^2 h'$.

Hence, condition that collision takes place between x^{th} and $(x+1)^{\text{th}}$ impact is $h>(2x-1)^2h'$ and $<(2x+1)^2h'$. If T be the time between x^{th} impact and the collision, we have $T=\frac{h-(2x-1)^2h'}{2x(2gh')^{\frac{1}{2}}}$; therefore height above plane $= T(2gh')^{\frac{1}{2}}-\frac{1}{2}gT^2 =$ &c., as in the Question.

6590. (By E. B. ELLIOTT, M.A.)—If there be any distribution of mass M in space, if I_a, I_b, I_c be its moments of inertia with regard to any three parallel lines A, B, C, and if a, b, c be the distances between the lines B and C, C and A, and A and B respectively; prove that the moment of inertia about any fourth parallel line may be expressed in the form

$$xI_a+yI_b+zI_c-M(a^2yz+b^2zx+c^2xy),$$

x, y, z being coordinates whose sum is unity of the fourth line with regard to the prism formed by the three.

Solution by J. W. RUSSELL, M.A.; *the* Rev. T. R. TERRY, M.A.; *and others.*

Let a plane through O, the centre of gravity of the mass, cut at right angles the three given lines in ABC, and the fourth line in P. Then, if (xyz) be the areal coordinates of P, and $(x'y'z')$ those of O referred to the triangle ABC, and I, I_0 the moments of inertia about parallel axes through P and O respectively, we have

$$\begin{aligned}
I &= I_0-M\left\{a^2(y-y')(z-z')+b^2(z-z')(x-x')+c^2(x-x')(y-y')\right\},\\
I_a &= I_0-M\left\{a^2y'z' \qquad\qquad +b^2(-z')(1-x')+c^2(-x')(1-y')\right\},\\
I_b &= I_0-M\left\{a^2(1-y')(-z')+b^2z'x' \qquad\qquad +c^2(-x')(1-y')\right\},\\
I_c &= I_0-M\left\{a^2(-y')(1-z')+b^2(1-z')(-x')+c^2x'y'\right\};
\end{aligned}$$

whence, obviously, $I=xI_a+yI_b+zI_c-M(a^2yz+b^2zx+c^2xy)$.

6325, 6367, 6501, 6562. (By R. KNOWLES, B.A., L.C.P.; Rev. D. THOMAS, M.A.; and others.)—In Quest. 6292, prove that

$$p_1 - 2p_2 + \dots + (n-1)(-)^{n-2} p_{n-1} = 0 \quad \dots\dots\dots\dots(1);$$

$$\tfrac{1}{2}p_1 - \tfrac{1}{3}p_2 + \dots + \frac{1}{n+1}(-)^{n-1} p_n = \frac{n}{n+1} \quad \dots\dots\dots\dots(2);$$

$$\tfrac{1}{3}p_1 - \tfrac{1}{4}p_2 + \dots + \frac{1}{n+2}(-1)^{n-1} p_n = \frac{n(n+3)}{2(n+1)(n+2)} \quad \dots\dots\dots(3);$$

$$1.2p_1 - 2.3p_2 + \dots + (n-1)\,n(-1)^{n-2} p_{n-1} = 0 \quad \dots\dots\dots(4);$$

$$\frac{p_1}{2.3} - \frac{p_2}{3.4} + \dots + \frac{1}{(n+1)(n+2)}(-1)^{n-1} p_n = \frac{n}{2(n+2)} \quad \dots\dots(5);$$

also, if p_r denote the coefficient x^r in the expansion of $(1+x)^n$, where n is a positive integer; and if m be also any positive integer, then

$$\frac{p_0}{m+1} - \frac{p_1}{m+2} + \dots + \frac{(-1)^n p_n}{m+n+1} = \frac{n(n-1)(n-2)\dots 1}{(m+1)(m+2)\dots(m+n+1)} \quad \dots\dots(6).$$

Solution by J. J. WALKER, M.A.

Considering the coefficient of x^{n+m+1} in the development of

$$u = (1+x)^n \log_e(1+x) = (p_0 x^n + p_1 x^{n-1} + \dots + p_n)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right),$$

which is $\quad (-)^m F(m+1) = (-1)^m \left(\dfrac{p_0}{m+1} - \dfrac{p_1}{m+2} \dots \dfrac{(-)^n p_n}{m+n+1}\right),$

and equating it with

$$\frac{1}{1.2\dots n+m+1}\,\frac{d^{n+m+1}u}{dx^{n+m+1}}, \quad \text{or} \quad \frac{(-1)^m}{1.2\dots(n+m+1)}\,\frac{n.n-1\dots 1.1.2\dots m}{(1+x)^{m+1}},$$

after making $x = 0$, the equality in 6562 is proved; and the first equality in 6325 is the particular case of the above, in which $m = 0$, since $\dfrac{n}{n+1} = p_0 - \dfrac{1}{n+1}$; that in 6367 is the case of $m = 1$.

Again, $F(m) - F(m+1)$ gives

$$\frac{p_0}{m(m+1)} - \dots + \dots (-1)^n \frac{p_n}{(m+n)(m+n+1)} = \frac{(n+1)\,n\dots 2.1}{m(m+1)\dots(m+n+1)},$$

which is a generalization of the second equality in 6501, the latter being the particular case of $m = 1$, since $\dfrac{n}{2(n+2)} = \dfrac{p_0}{1.2} - \dfrac{1}{n+2}$.

To verify the first equality of Quest. 6325, consider the coefficient of $-x^{n-1}$ in the development of $u^n \dfrac{du^{-1}}{dx}$, where $u = 1+x$, that is, in

$$(x^n + p_1 x^{n-1} + \dots)(-1 + 2x - 3x^2 + \dots).$$

It is $p_1 - 2p_2 + 3p_3 - \dots$. But $u^n \dfrac{du^{-}}{dx}$ also $= -(1+x)^{n-2}$, containing no power of x higher than $n-2$.

Again, consider $u^n \frac{d^2 u^{-1}}{dx^2}$. The coefficient of x^{n-1} in the form

$$(x^n + p_1 x^{n-1} + \dots)(1.2 - 2.3x + 3.4x^2 - 4.5x^3 + \dots)$$

is the series of the first equality in Quest. 6501. But $u^n \frac{d^2 u^{-1}}{dx^2}$ also equals $2(1+x)^{n-3}$, and this contains no higher power of x than x^{n-3}.

By considering, similarly, $u^n \frac{d^r u^{-1}}{dx^r}$, where r is any integer less than n, we obtain the more general results

$$1.2 \dots rp_1 - 2.3 \dots (r+1)p_2 + \dots\dots (-1)^n n(n+1) \dots\dots (n+r-1)p_n = 0,$$
$$1.2 \dots rp_2 - 2.3 \dots (r+1)p_3 + \dots (-1)^{n-1}(n-1)n \dots (n+r-2)p_{n-1} = 0,$$

and so on.

6579. (By J. J. WALKER, M.A.)—If P, P′ are two points on an ellipse, the foci of which are S, H; and if X, X′, Y are any points in the productions of SP, SP′, PP′ respectively; prove that

$$\cos \tfrac{1}{2}(\text{XPY} + \text{X}'\text{P}'\text{Y}) : \cos \tfrac{1}{2}(\text{HPY} + \text{HP}'\text{Y}) = \cos \tfrac{1}{2}\text{PSP}' : \cos \tfrac{1}{2}\text{PHP}'.$$

Solution by G. TURRIFF, M.A.; *the* Rev. T. R. TERRY, M.A.; *and others.*

Since $$\frac{\text{SP}' - \text{SP}}{\text{PP}'} = \frac{\text{HP} - \text{HP}'}{\text{PP}'},$$

$$\frac{\sin \text{XPY} - \sin \text{X}'\text{P}'\text{Y}}{\sin \text{PSP}'} = \frac{\sin \text{HP}'\text{Y} - \sin \text{HPY}}{\sin \text{PHP}'};$$

therefore $$\frac{\cos \tfrac{1}{2}(\text{XPY} + \text{X}'\text{P}'\text{Y})}{\cos \tfrac{1}{2}\text{PSP}'} = \frac{\cos \tfrac{1}{2}(\text{HPY} + \text{HP}'\text{Y})}{\cos \tfrac{1}{2}\text{PHP}'}.$$

[In the limit, when P′ coincides with P, this gives cos XPY = cos HPY; therefore ∠XPY = HPY, a well-known result. The same proof establishes the property for the hyperbola, by interchanging HP, HP′, and writing S for X.]

5433. (By Prof. PRATT, M.A.)—Given $u = \text{F}(y)$, $y = \text{F}'\{z + xvf(y)\}$, $v = \text{F}_1 t$, $t = \text{F}_2\{z + xvf_1(t)\}$; expand u in a series of ascending, positive, integral powers of x (x not being a function of z).

Solution by J. J. WALKER, M.A.

As F′ does not appear to be used in the sense for which it is very convenient to reserve it, viz., the first derived of F, I will substitute F_0 for F′. Then, v being a function of x and z,

whence $\quad \dfrac{dy}{dx} = \mathrm{F}'_0\{z+\dots\}\left\{vf(y)+x\dfrac{dv}{dx}f(y)+xvf'(y)\dfrac{dy}{dx}\right\},$

$$\frac{dy}{dx} = \left[\left(v+x\frac{dv}{dx}\right)f(y)\,\mathrm{F}'_0\{z+xvf(y)\}\right] : \left[1-xvf'(y)\,\mathrm{F}'_0\{z+\dots\}\right].$$

Again, $\quad \dfrac{dy}{dz} = \mathrm{F}'_0\{z+\dots\}\left\{1+x\dfrac{dv}{dz}f(y)+xvf'(y)\dfrac{dy}{dz}\right\},$ whence

$$\frac{dy}{dz} = \left[\left\{1+x\frac{dv}{dz}f(y)\right\}\mathrm{F}'_0\{z+\dots\}\right] : \left[1-xvf'(y)\,\mathrm{F}'_0\{z+\dots\}\right];$$

so that $\quad \dfrac{dy}{dx} = \left[\left(v+x\dfrac{dv}{dx}\right)f(y) : \left\{1+x\dfrac{dv}{dz}f(y)\right\}\right]\dfrac{dy}{dz}.$

Following the steps of the proof of LAGRANGE's theorem, the general formula $\dfrac{d^nu}{dx^n} = \dfrac{d^{n-1}}{dz^{n-1}}\left[\left\{\left(v+x\dfrac{dv}{dx}\right)f(y)\right\}^n\dfrac{du}{dz} : \left\{1+x\dfrac{dv}{dz}f(y)\right\}^n\right]$ is obtained; so that, when $x=0$,

$$\frac{d^nu}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left[\{vf(y)\}^n\frac{d\,\mathrm{F}\{\mathrm{F}_0(z)\}}{dz}\right],$$

wherein $y=\mathrm{F}_0(z)$, $v=\mathrm{F}_1\{\mathrm{F}_2(z)\}$; and the required expansion is, restoring F' in place of F_0,

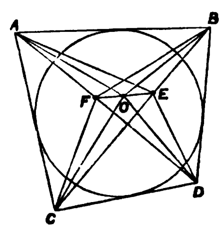
$$u = \mathrm{F}\{\mathrm{F}'(z)\}+x\,\mathrm{F}_1\{\mathrm{F}_2(z)\}f\{\mathrm{F}'(z)\}\frac{d\,\mathrm{F}\{\mathrm{F}'(z)\}}{dz}+\frac{x^2}{1.2}\frac{d}{dz}$$

$$\times\left[\left(\mathrm{F}_1\{\mathrm{F}_2(z)\}f\{\mathrm{F}'(z)\}\right)^2\frac{d\,\mathrm{F}\{\mathrm{F}'(z)\}}{dz}\right]$$

$$+\frac{x^3}{1.2.3}\frac{d^2}{dz^2}\left[(\dots)^3\frac{d\,\mathrm{F}\{\mathrm{F}'(z)\}}{dz}\right]+\dots.$$

6436. (By D. EDWARDES.)—If a quadrilateral be circumscribed about a circle, prove that the middle points of its diagonals and the centre of the circle lie in a straight line.

Solution by H. MURPHY; J. O'REGAN; *and others.*



Let E, F be the middles of the diagonals, and O the centre; then $\triangle$AFC and $\triangle$BFD = ½ quad. ABDC = $\triangle$AEC + $\triangle$BED = $\triangle$AOC + $\triangle$BOD; hence we have the given bases AC and BD, and the sum of the areas of two triangles having a common vertex; thus the locus of vertices is a straight line EOF.

[This question is a particular case of CHASLES' theorem, that the locus of the centres of conics touching the same four straight lines is a straight line.]

6596. (By Professor SYLVESTER, F.R.S.)—APB is a semi-ellipse of which S is the focus, and AB the major axis; and AMB is a semicircle drawn on AB; show (1) that if two bodies, attracted each towards its own centre of force at S, and leaving the point A at the same moment, move in these two curves, the one will be at P when the other is at M, PM being any ordinate perpendicular to AB; and (2) state also the theorem or the case when S coincides with A.

Solution by J. J. WALKER, M.A.

1. What is stated in the Question will be the case if a certain relation hold between the elliptic and circular forces, which may be most neatly expressed as that the accelerations towards S at A must be equal one to the other. For the elliptic movement, if f_1 is the acceleration at A, then, generally, at P, $f = \dfrac{f_1\,\mathrm{SA}^2}{\mathrm{SP}^2} = \dfrac{ah^2}{b^2\,\mathrm{SP}^2}$; $\therefore f_1 = \dfrac{ah^2}{b^2 . \mathrm{SA}^2}$.

Again, for the circular movement, $f' = \dfrac{2h'^2\mathrm{AB}}{\mathrm{MM}'^3 . \mathrm{SM}^2}$, MM′ being the chord through S; and, at A, $f_1' = \dfrac{h'^2}{a . \mathrm{SA}^2}$. Hence $\dfrac{f_1}{f_1'} = \dfrac{a^2h^2}{b^2h'^2}$; and, for the condition of the Quest., obviously, $h^2 : h'^2 = b^2 : a^2$, so that $f_1 = f_1'$.

2. The relation between the forces, in this case, may be stated by the condition that the accelerations at B towards A must be equal. For, generally, in the elliptic movement, $f = \dfrac{h^2\mathrm{CG}^3}{a^2b^2\mathrm{AP}^2}$, where CG is the length of a line drawn from the centre (C) to the tangent at P, parallel to SP, so that, at B, $f_2 = \dfrac{h^2}{4ab^2}$. In the circle, $f' = \dfrac{2h'^2(2a)^2}{\mathrm{AM}^5}$ generally; and, in particular, at B, $f_2' = \dfrac{h'^2}{4a^3}$, so that $\dfrac{f_2}{f_2'} = \dfrac{a^2h^2}{b^2h'^2}$, and therefore $f_2 = f_2'$. I understand the centre of force to be removed to A, the ellipse remaining unchanged, or not.

[Prof. SYLVESTER remarks that one force varies as d^{-2} and the other according to the law given by NEWTON, viz., $d^{-2}q^{-3}$ (q being the complete chord of which d is a part), but these limits need not be stated in the Question. When S coincides with A, the ellipse becomes a straight line, and B becomes the *starting* point of the body moving in this line, acted on by a force varying as d^{-2}; also the force in the circle then varies as d^{-5}.]

6581. (By W. R. WESTROPP ROBERTS, M.A.)—Through a fixed chord AB of a twisted cubic are drawn two planes harmonically conjugate with two fixed planes through the same chord. Show that, if the variable planes meet the cubic in two points P and Q, the chord PQ meets two fixed lines.

Solution by Professor TOWNSEND, M.A., F.R.S.

Denoting by U and V the two quadric cones, having A and B for vertices which intersect in the cubic, and in the line AB; by E and F the

two thirds points of intersection with the curve of the two fixed planes through AB, and by M and N the two unique lines of intersection of the two pairs of tangent planes to U and V along their two pairs of generating lines AE and AF, BE and BF, respectively; then, the two variable planes APB and AQB dividing harmonically in every position the angle between the two fixed planes AEB and AFB, and the two variable planes PAQ and PBQ passing consequently in every position, the former through the fixed line M and the latter through the fixed line N, the variable line PQ common to the two latter planes passes consequently in every position through the two fixed lines M and N; and therefore, &c.

If the two variable planes APB and AQB, in place of dividing harmonically in every position the angle between the two fixed planes AEB and AFB, divided it more generally in any constant anharmonic ratio; since then the two variable planes PAQ and PBQ, instead of passing respectively through the two fixed lines M and N, would touch respectively two fixed quadric cones U′ and V′ having double contact with U and V along the two pairs of lines AE and AF, BE and BF, respectively, and breaking up into the two pairs of planes touching them along those lines in the case of harmonic section, the variable line PQ, instead of passing in every position through the two fixed lines M and N, would touch in every position the two fixed cones U′ and V′, and would pass through M and N in the case only of harmonic section when the two cones break up into pairs of planes intersecting in those lines.

That, in the same case, the variable chord PQ generates a ruled quadric, passing through the cubic and through the two lines M and N, may be readily shown as follows. The ruled quadric determined by any three positions P_1Q_1, P_2Q_2, P_3Q_3 of PQ, passing through the two lines M and N, and therefore through the two points A and B, intersects consequently the cubic in eight points P_1 and Q_1, P_2 and Q_2, P_3 and Q_3, A and B, and therefore contains it altogether; and every generator of it L of the system opposite to M and N, intersecting in consequence the curve in a pair of points P and Q connecting through M and N, coincides accordingly with a position of PQ; and therefore, &c.

When the two fixed planes AEB and AFB divide, as they may, harmonically the angle between the two planes through AB which touch the curve at A and B, then is AB a position of PQ, but otherwise not. In that case, therefore, the quadric generated by PQ passes through the complete intersection of the two cones U and V, while in all other cases it passes only through the curve, but not through the line common to them both.

6526. (By the Editor.)—From the points in which any two conjugate diameters of an ellipse meet a fixed tangent, tangents are drawn to a fixed confocal; prove that the locus of their intersection is a circle whose centre is on the normal to the fixed tangent.

Solution by C. Taylor, M.A.

From the points in which any two parallel tangents or conjugate diameters of an ellipse meet the tangent at a fixed point O, draw four tangents to a fixed confocal. In any assumed position the other four inter-

sections of these tangents lie in a circle. Consider this circle fixed, and let another quadrilateral be inscribed in it, so as to envelop the inner confocal. The opposite sides of this quadrilateral intersect at two points on the tangent at O; and it is evident that the second tangents from these two points to the original ellipse are parallel, and conversely. Hence the required locus is a circle. Making one of the parallel tangents coincide with the tangent at O, we see that the centre of the circle lies on the normal at O. [See also Question 2911, whereof an algebraical proof is given on pp. 31—33 of Vol. XIII. of the *Reprint*.]

NOTE ON PROF. CROFTON'S QUESTION 4795. *By* PROF. NASH, M.A.

The solution of this Question suggests the following query:— The theorem is its own inverse with respect to each of the foci, the constant of inversion being so chosen that the other foci are inverse points. Is this a sufficient proof of the truth of the theorem? I have frequently found that theorems repeat themselves in this way when inverted or reciprocated. For example, Euclid VI. D is its own inverse with respect to any point.

As examples in reciprocation, Quest. 1471 in Prof. WOLSTENHOLME'S *Book of Problems* may be taken, and also the theorem that the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus. In both of these cases only one point can be taken as origin of reciprocation, which is not generally the case with inversion.

6606. (By ELIZABETH BLACKWOOD.)—In the integral

$$\int dw \int dx \int dy \int dz \; \phi(w, x, y, z),$$

the variables are each between the limits a and $-a$, and the integration is further restricted by the condition that $(wx-yz)$ is positive; determine the limits of integration.

Solution by W. B. GROVE, B.A.

Let $Q \equiv w_{1'.2}\, x_{1'.2}\, y_{1'.2}\, z_{1'.2}\, p\,(wx-yz)$, where the symbols have the meanings attached to them in Mr. McCOLL'S first paper in *Proc. Lond. Math. Soc.* Then it will be found that

$$Q = (w_{1'.0}\, x_{1'.0} + w_{0'.2}\, x_{0'.2})\,(y_{1'.3}\, z_{3'.2} + y_{3'.4}\, z_{1'.2} + y_{4'.2}\, z_{1'.3})$$
$$+ (w_{1'.0}\, x_{0'.2} + w_{0'.2}\, x_{1'.0})\,(y_{1'.4}\, z_{3'.2} + y_{3'.2}\, z_{1'.3}).$$

Hence we infer that the limits of integration, which in this statement are denoted by suffixes, are as in the annexed table :—

$w_1 = a$	$x_1 = a$	$y_1 = a$	$z_1 = a$
$w_2 = -a$	$x_2 = -a$	$y_2 = -a$	$z_2 = -a$
		$y_3 = \dfrac{wx}{a}$	$z_3 = \dfrac{wx}{y}$
		$y_4 = -\dfrac{wx}{a}$	

6569. (By J. R. Harris, M.A.) — Two trains, of lengths m, n respectively, are at distances a, b from a level crossing, towards which they are moving with velocities that are equally likely to be any possible magnitude; prove that, according as the tail of the second train is nearer to or further from the crossing than the head of the first train, the chance of an accident is $(a > b)$

$$\tfrac{1}{2}\frac{an+bm+mn}{a(a+m)}, \text{ or } 1-\tfrac{1}{2}\frac{a^2+b^2+am+bn}{(a+m)(b+n)}.$$

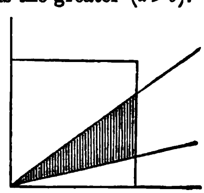
Solution by G. F. Walker, M.A.; Prof. Matz, M.A.; *and others.*

Let x, y be the velocities of the two trains, and suppose that neither exceeds l: there will be an accident if the ratio $x : y$ lie anywhere between the ratios $a+m : b$, $a : b+n$, of which the former is the greater $(a > b)$.

Take two rectangular axes of x and y and draw the lines

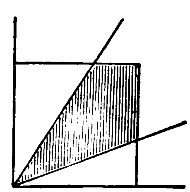
$$x = l, \quad y = l, \quad y = \frac{bx}{a+m}, \quad y = \frac{b+n}{a}x.$$

Then we will have the figures (1) and (2) in the cases given, and the chance is in each case the ratio of the shaded area to that of the square, and is



(Fig. 1.)

Fig. (1)
$$\frac{\tfrac{1}{2}l\left[l\dfrac{b+n}{a}-l\dfrac{b}{a+m}\right]}{l^2} = \tfrac{1}{2}\frac{an+bm+mn}{a(a+m)},$$



(Fig. 2.)

Fig. (2)
$$\frac{l^2-\tfrac{1}{2}l\left[l\dfrac{a}{b+n}+l\dfrac{b}{a+m}\right]}{l^2} = 1-\tfrac{1}{2}\frac{a^2+b^2+am+bn}{(a+m)(b+n)}.$$

6599. (By Professor MINCHIN, M.A.)—At any two points, A and B, in the plane of any lamina, draw lines AO and BO equal to the radii of gyration of the lamina round A and B respectively, thus constructing a triangle ABO; then prove that the radius of gyration round any point P in base AB is PO.

Solution by E. B. ELLIOTT, M.A.; W. M. COATES, B.A.; *and others.*

This is a simple case of Quest. 6590. It may also be briefly proved independently as follows:—

If M be the position of an element δm of the lamina, we have

$$MP^2 = \frac{MA^2 . PB + MB^2 . AP}{AB} - AP . PB;$$

multiplying which by δm, and summing over the whole lamina, we get

$$I_p = \frac{I_a . PB + I_b . AP}{AB} - mAP . PB;$$

or, dividing by m, $\quad k^2 = \dfrac{OA^2 . PB + OB^2 . AP}{AB} - AP . PB = OP^2.$

6617. (By the Rev. W. A. WHITWORTH, M.A.)—A row of n letters may be permuted by moving any letter backwards or forwards *over the next two letters*. Show that, by continually repeating this operation, the n letters can be brought into $\frac{1}{2}n\,!$ different orders; that is, exactly half of the possible permutations of the n letters can be formed.

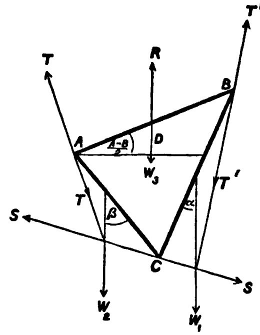
Solution by J. A. KEALY, M.A.; W. B. GROVE, B.A.; *and others.*

In any row of n letters, it is obvious that any letter may be brought into the first place by the method described. Suppose, then, that the theorem in question is true for $(n-1)$ letters; then we have $\frac{1}{2}(n-1)\,!$ permutations of these n letters in which a particular one stands first. Therefore the total number of permutations is $\frac{1}{2}n . (n-1)\,! = \frac{1}{2} . n\,!$; *i.e.*, if the theorem is true for $(n-1)$ letters, it is true for n letters; but it is evidently true for three letters, etc.

6236. (By W. H. H. HUDSON, M.A.)—A triangle ABC, formed of three rods jointed together, is supported by a rough peg under the middle point of AB; prove (1) that the least angle of friction is $\frac{1}{2}(A-B)$, (2) that the sides AC, BC are equally inclined to the vertical, (3) that the strain at the joint C is equally inclined to the horizon with the side AB, and (4) obtain also the magnitude of this and of the other strains.

Solution by D. EDWARDES; Prof. DROZ, M.A.; *and others.*

(1). Supposing the bars to be of the same thickness and density; then, since the centroid of the perimeter of a triangle is the centre of the circle inscribed in the triangle formed by joining the middle points of the sides, if G be this centre, D the middle point of AB, we have $\angle ADG = 90^\circ - \frac{1}{2}(A-B)$. Now, for the equilibrium of the system, the total resistance at D must obviously be vertical, and act along GD, which it can only do provided λ be not less than $90^\circ - \angle ADG$. The least value of λ is therefore $\frac{1}{2}(A-B)$.



(2). The inclination of AB to the horizon is evidently $\frac{1}{2}(A-B)$, hence the inclination (β) of AC to the horizon is $90^\circ - \frac{1}{2}(A+B) = \frac{1}{2}C$. Hence $\alpha = \beta = \frac{1}{2}C$.

(3). Let S be the strain at C, making an angle θ with the horizon. Taking moments about A, B for the bars AC, BC respectively, we have

$$2S\sin\left[\tfrac{1}{2}(A+B)-\theta\right] = W_2\sin\tfrac{1}{2}C,\quad 2S\sin\left[\tfrac{1}{2}(A+B)+\theta\right] = W_1\sin\tfrac{1}{2}C;$$

whence $$\frac{\sin\left[\frac{1}{2}(A+B)-\theta\right]}{\sin\left[\frac{1}{2}(A+B)+\theta\right]} = \frac{W_2}{W_1} = \frac{\sin B}{\sin A};\quad \therefore\ \theta = \tfrac{1}{2}(A-B).$$

(4). If $$\frac{W_1}{\sin A} = \frac{W_2}{\sin B} = \frac{W_3}{\sin C} = k,$$

then $$2S\sin B = W_2\sin\tfrac{1}{2}C;\quad \therefore\ S = \tfrac{1}{2}k\sin\tfrac{1}{2}C.$$

If T, T′ be the strains at A and B,

$$T = \left\{W_2^2 + S^2 + 2W_2S\sin\tfrac{1}{2}(A-B)\right\}^{\frac{1}{2}}$$
$$= \tfrac{1}{2}k\left\{\sin^2\tfrac{1}{2}C + 2\sin B(\sin A+\sin B)\right\}^{\frac{1}{2}},$$
$$T' = \left\{W_1^2 + S^2 - 2W_1S\sin\tfrac{1}{2}(A-B)\right\}^{\frac{1}{2}}$$
$$= \tfrac{1}{2}k\left\{\sin^2\tfrac{1}{2}C + 2\sin A(\sin A+\sin B)\right\}^{\frac{1}{2}}.$$

6123. (By Professor BURNSIDE, M.A.)—If the sides *ad*, *bd*, *cd* of a tetrahedron *abcd* meet a surface of the second order in the points a, a_1; b, b_1; c, c_1, respectively; prove that the sections of the surface by the planes bca_1, cab_1, abc_1 are all touched by another plane section of the surface.

Solution by W. J. C. SHARP, M.A.; CHRISTINE LADD; *and others.*

If *bcd*, *cda*, *dab*, *abc* be the planes of reference, the equation to the quadric is $Dw^2 + 2Fyz + 2Gzx + 2Hxy + 2Lxw + 2Myw + 2Nzw = 0$, and the planes a_1bc, ab_1c, abc_1, abc are

$$Dw + 2Lx = 0,\ Dw + 2My = 0,\ Dw + 2Nz = 0,\ \text{and}\ w = 0 \ldots (1),$$

the intersections of which with the plane

$$Dw + 2Lx + 2My + 2Nz = 0 \text{ .... } (2)$$

touch the section of the quadric by that plane.

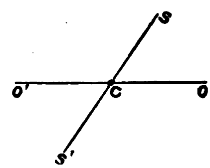
For the equation to the tangent plane at $(x_1 y_1 z_1 w_1)$ reduces to

$$(Gz_1 + Hy_1)\,x + (Fz_1 + Hx_1)\,y + (Fy_1 + Gx_1)\,z = 0,$$

(the tangent plane to the quadric $Fyz + Gzx + Hxy = 0$, which passes through the section,) where the equation (2) and any one of the equations (1) are satisfied by $(xyzw)$ and $(x_1 y_1 z_1 w_1)$.

6538. (By Professor GENESE, M.A.)—Find the envelop of the latera recta of conics having a given director circle, and passing through a given point.

Solution by Professor WOLSTENHOLME, M.A.



Let C be the centre of the director circle, O the given point, then C will be the centre of the ellipse, and if OO′ be bisected in C, OO′ will be a diameter of each of the conics; and if S, S′ be the real foci of one of them, OS . O′S = SO . S′O = excess of square of radius of director circle above CO^2. Hence S, S′ lie on a Cassinian whose foci are O, O′, and the envelop of the latus rectum is the first negative pedal of a Cassinian with respect to its centre, which is a rectangular hyperbola, and the Cassinian is Bernoulli's Lemniscate when the radius of the director circle $= \sqrt{2}\,.\,CO$.

5574. (By R. TUCKER, M.A.)—Three particles are projected simultaneously in the same vertical plane, with velocities v_1, v_2, v_3, at inclinations $\alpha_1, \alpha_2, \alpha_3$ to the horizon; show that, when their directions are parallel,

$$\frac{\sin(\alpha_2 - \alpha_3)}{v_1} + \frac{\sin(\alpha_3 - \alpha_1)}{v_2} + \frac{\sin(\alpha_1 - \alpha_2)}{v_3} = 0.$$

Solution by D. EDWARDES; J. H. TURRELL, M.A.; *and others.*

The vertical velocity of particle (α_1) after time t is $v_1 \sin\alpha_1 - gt$, and its horizontal velocity is $v_1 \cos\alpha$. Let θ be the inclination to horizon of their directions when parallel, then

$$\tan\theta = \frac{v_1 \sin\alpha_1 - gt}{v_1 \cos\alpha_1} = \frac{v_2 \sin\alpha_2 - gt}{v_2 \cos\alpha_2} = \frac{v_3 \sin\alpha_3 - gt}{v_3 \cos\alpha_3};$$

therefore $$\frac{v_1 \sin\alpha_1 - v_2 \sin\alpha_2}{v_1 \cos\alpha_1 - v_2 \cos\alpha_2} = \frac{v_2 \sin\alpha_2 - v_3 \sin\alpha_3}{v_2 \cos\alpha_2 - v_3 \cos\alpha_3};$$

whence, simplifying, we get the required result.

5918. (By Prof. SEITZ, M.A.)—Prove that the average distance of all points within an ellipse from the extremity of the major axis is

$$\frac{2b(1+2e^2)}{3\pi e^2} - \frac{2a(1-4e^2)}{3\pi e^3}\sin^{-1}e.$$

Solution by J. HAMMOND, M.A.; ELIZABETH BLACKWOOD; *and others.*

The average distance is $\quad \frac{2}{\pi ab}\iint r\,.\,r\,dr\,d\theta,$

or $\quad \frac{2}{3\pi ab}\int_0^{\frac{1}{2}\pi} r^3 d\theta,$ where $r\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) = \frac{2\cos\theta}{a}$ .... (1).

Put $e = \sin\alpha,\ b = a\cos\alpha,$ then (1) gives $r = \frac{2a\cos\theta}{(1+\tan^2\alpha\sin^2\theta)}.$

Let $\tan\alpha\sin\theta = \tan\phi,$ then $\tan\alpha\cos\theta\,d\theta = \sec^2\phi\,d\phi,$ and $r = 2a\cos\theta\cos^2\phi$; therefore the average distance is

$$\frac{16a^2}{3\pi b}\int_0^{\alpha}\cos^4\phi\left(1-\frac{\tan^2\phi}{\tan^2\alpha}\right)\cot\alpha\,d\phi$$

$$= \frac{16a^2\cos\alpha}{3\pi b\sin^3\alpha}\int_0^{\alpha}\cos^2\phi\sin(\alpha+\phi)\sin(\alpha-\phi)\,d\phi$$

$$= \frac{2a}{3\pi e^3}\int_0^{\alpha} 2(1+\cos 2\phi)(\cos 2\phi-\cos 2\alpha)\,d\phi$$

$$= \frac{2a}{3\pi e^3}\int_0^{\alpha}\left\{\cos 4\phi + 2\cos 2\phi(1-\cos 2\alpha)+(1-2\cos 2\alpha)\right\}d\phi$$

$$= \frac{2a}{3\pi e^3}\left\{\frac{\sin 4\alpha}{4}+\sin 2\alpha(1-\cos 2\alpha)+\alpha(1-2\cos 2\alpha)\right\}.$$

Now $\sin\alpha = e,\ \cos 2\alpha = 1-2e^2,\ \sin 2\alpha = 2e\cos\alpha,\ a\cos\alpha = b$; hence the average distance is that given in the question.

6553. (By G. F. WALKER, M.A.)—Solve the equations

$$x^2(y+z) = a^3,\quad y^2(z+x) = b^3,\quad z^2(x+y) = c^3.$$

Solution by Rev. J. L. KITCHIN, M.A.; BELLE EASTON; *and others.*

Multiplying, and also adding, the equations, we have

$$x^2y^2z^2(x^2y+xy^2+xz^2+x^2z+yz^2+y^2z+2xyz) = a^3b^3c^3,$$

$$x^2y+xy^2+xz^2+x^2z+y^2z+yz^2 = a^3+b^3+c^3;$$

therefore $\quad 2x^3y^3z^3+(a^3+b^3+c^3)\,x^2y^2z^2 = a^3b^3c^3,$

a cubic for finding xyz. Let t be a root, then $xyz = t$, therefore

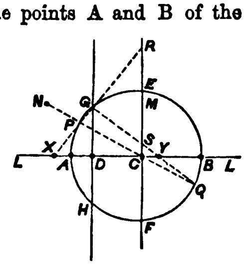
$$\frac{1}{y}+\frac{1}{z} = \frac{a^3}{tx},\ \text{\&c. \&c.},$$

three ordinary equations for finding $x,\ y,\ z$.

5422. (By Professor Townsend, F.R.S.)—Four points A, B, C, D being supposed given or taken arbitrarily on a straight line L; construct, by elementary geometry, the two X and Y, on the line, the distance of each of which from one of the four D shall be the harmonic mean of its distances from the remaining three A, B, C.

Solution by the Proposer.

On the interval AB, between the two extreme points A and B of the given or assumed tetrad A, B, C, D, as diameter, describe a circle, as in the figure, intersecting the two perpendiculars to the given line L at the remaining two points C and D of the tetrad in the two pairs of points E and F, G and H, respectively; and connect either point G of the latter pair with the two required points X and Y, supposed found, on L; the connecting lines GX and GY intersecting the circle at the two second points P and Q, and the perpendicular to L through C at the two points R and S; then since, by



hypothesis, $\quad AX^{-1}+BX^{-1}+CX^{-1} = 3 \,.\, DX^{-1}$,

and $\quad AY^{-1}+BY^{-1}+CY^{-1} = 3 \,.\, DY^{-1}$,

therefore, by the theory of poles and polars,

$GX^{-1}+PX^{-1}+RX^{-1} = 3 \,.\, GX^{-1}$, and $GY^{-1}+QY^{-1}+SY^{-1} = 3 \,.\, GY^{-1}$,

therefore $\quad PX^{-1}+RX^{-1} = 2 \,.\, GX^{-1}$, and $QY^{-1}+SY^{-1} = 2 \,.\, GY^{-1}$,

and therefore GX is the harmonic mean between PX and RX, and GY the harmonic mean between QY and SY; consequently the pencil of four rays which subtends from C the two tetrads of points PRGX and QSGY is harmonic, and therefore GM = GN, where MN is the parallel through G to XY or L. Hence the point N is given, and therefore the two P and Q, and therefore the two X and Y, and therefore, &c.

Denoting by a, b, c the distances of A, B, C, and by x and y those of X and Y from D, it follows at once, from the algebraic solution of the same problem, that the latter are connected with the former by the relations $x+y = 2\,\dfrac{bc+ca+ab}{a+b+c}$, and $xy = 3\,\dfrac{abc}{a+b+c}$; which accordingly, regard of course being had to the signs as well as to the magnitudes of the several quantities involved, are the formulæ by which to calculate in numbers the distances of X and Y from D when those of A, B, C from it are given or known.

6591. (By W. H. H. Hudson, M.A.)—A man standing at the North Pole whirls 24 lbs. Troy weight on a smooth horizontal plane, by a string a yard long, at the rate of 100 turns a minute, and finds that the difference of the forces he has to exert, according as he whirls it one way or the opposite, is roughly 39 grains: find the period of the earth's rotation.

Solution by Rev. T. R. Terry, M.A.; Prof. Matz, M.A.; *and others.*

Let n be the number of turns made per minute by the earth. Then, if we take a foot and a second as the units, we have

$$3\left[\left\{\frac{(100+n)2\pi}{60}\right\}^2-\left\{\frac{(100-n)2\pi}{60}\right\}^2\right]=\frac{39g}{24\times5760};$$

$$\therefore\ \text{approx. } n=\frac{13g}{61440\pi^2},\quad \therefore\ \text{number of turns per day}=\frac{39g}{128\pi^2}.$$

Put $g=32$ and $\pi^2=9{\cdot}9$, and we get, approximately, number of turns per day $=\cdot996$. Period of earth's rotation $=1$ day 6 min.

6254. (By J. J. Walker, M.A.)—Find at what points on an ellipse the continued product of the semi-diameter, the perpendicular on tangent, and the cosecant of the angle between them, is a minimum.

Solution by R. Knowles, B.A., L.C.P.; G. Eastwood, M.A.; *and others.*

$$\text{The product}=r^3\,.\,\frac{d\theta}{dr}=\pm\frac{b^2}{c^2\cos\theta\,.\,\sin\theta}=u;$$

$$\frac{du}{d\theta}=\mp\frac{b^2}{c^2}(\operatorname{cosec}^2\theta-\sec^2\theta)=0,\quad\therefore\ \tan\theta=\pm1;$$

$$\frac{d^2u}{d\theta^2}=\pm\frac{2b^2}{c^2}\left(\frac{\operatorname{cosec}^2\theta}{\tan\theta}+\tan\theta\,.\,\sec^2\theta\right);$$

therefore $\tan\theta=\pm1$ determines all the points on the curve for which (u) is a minimum.

6080. (By J. F. Moulton, M.A.)—A quantity of fluid fills a paraboloid of latus rectum c to a height h, the axis being vertical and vertex downwards. The density of the fluid varies as the depth. If the fluid pass into a vessel of the form generated round the axis of x, by the curve $a^4y^2=2ch^2x(a-x)(2a-x)$, where a is any constant, the density will vary as (depth)^2.

Solution by the Rev. T. R. Terry, M.A.; Lizzie A. Kittudge; *and others.*

Let large letters refer to old vessel, and small ones to the new. Then

$$\int_0^h\pi Y^2dX=\tfrac{1}{2}\pi ch^2=\int_0^a\pi y^2dx;$$

therefore fluid rises to height a in new vessel.

$$\text{Quantity below depth } hD=\int_0^{h(1-D)}\pi Y^2dX=\tfrac{1}{2}\pi ch^2(1-D)^2,$$

and density at the depth is μhD;

Quantity below depth ad is $\int_0^{a(1-d)} \pi y^2 dx = \frac{1}{2}\pi c h^2 (1-d^2)^2$;

therefore, equating these expressions, $D = d^2$; therefore the density at depth ad is $\mu h d^2$; that is, it varies as (depth)^2.

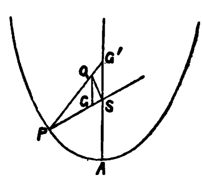
6603. (Professor GENESE, M.A.)—Given the base and vertical angle of a triangle, find the envelop of the nine-point circle.

Solution by W. M. COATES, B.A.; E. RUTTER; *and others.*

All triangles satisfying the given conditions have the same circumscribing circle, and the diameter of the nine-point circle of a triangle is equal to the radius of the circumscribing circle; moreover, all the nine-point circles pass through the middle point of the given base; consequently the envelop is a circle with the middle point of the base as centre, and radius equal to radius of circumscribing circle.

6512. (By Professor MATZ, M.A.)—A heavy prismatic bar of infinitesimal cross-section and length $2c$ rests against the concave arc of a vertical elliptic bowl and a pin placed at the focus; prove that the bar is in equilibrium when its inclination to the vertical is $\cos^{-1}\left\{\frac{1}{e}\left[\frac{b}{a^{\frac{1}{2}}c^{\frac{1}{2}}}-1\right]\right\}$, where (a, b) are the axes and e the eccentricity of the ellipse.

Solution by G. F. WALKER, M.A.; J. A. KEALY, M.A.; *and others.*



Let the reactions at P and S meet in Q, and let G be the centre of gravity; then QG must be vertical, and therefore parallel to the axis.

Let $\angle$ PSA $= \theta$, and $\angle$ SPQ $= \chi$. Since SG$' = e$.SP, we have $\sin\chi = e\sin(\theta-\chi)$, and $\tan\chi = \frac{e\sin\theta}{1+e\cos\theta}$; and if SP $= r$, since QSP is a right angle and QG is parallel to the axis,

$$(r-c)\sin\theta = (r\tan\chi)\cos\theta, \quad \therefore \left[1-\frac{c(1+e\cos\theta)}{a(1-e^2)}\right]\sin\theta = \frac{e\sin\theta}{1+e\cos\theta}.\cos\theta,$$

and putting $a(1-e^2) = \frac{b^2}{a}$, $\frac{ac(1+e\cos\theta)}{b^2} = \frac{1}{1+e\cos\theta}$; therefore, &c.

6622. (By T. H. Attwater, M.A.)—If the tangent to an ellipse at P meet the major axis in T, and the tangent at P′ meet the minor axis in T′, and if CP, CP′ are conjugate; prove that TT′ is parallel to an equi-conjugate diameter.

Solution by J. McDowell, M.A.; A. Anderson, B.A.; *and others.*

Let (x', y') be the coordinates of P, then $\dfrac{ay'}{b}, \dfrac{bx'}{a}$ will be those of P′; therefore $\text{CT} = \dfrac{a^2}{x'}$, $\text{CT}' = \dfrac{ab}{x'}$; therefore equation of diameter parallel to TT′ is $\dfrac{x}{a} + \dfrac{y}{b} = 0$; therefore square of this semi-diameter $= \frac{1}{2}(a^2+b^2)$.

6588. (By A. Martin, M.A.)—A person is to draw two from the tickets 1, 2, 3 ... 100, all tickets being equally likely. If both tickets be squares, he is to receive £100; if the lower, only £50; if the higher, only £20; find the worth of his chance of gain.

Solution by W. B. Grove, B.A.; Christine Ladd; *and others.*

There are 10 squares among the first hundred numbers; hence the number of ways in which two squares can be drawn $= \dfrac{10 \cdot 9}{1 \cdot 2} = 45$. The number of ways in which two numbers can be drawn, of which the lower only is a square, is $\sum\limits_{r=1}^{r=10} (90 - r^2 + r) = 570$.

The number of ways, if the higher only is a square, is $\sum\limits_{r=1}^{r=10} (r^2 - r) = 330$; and the total number of ways of drawing two tickets $= 4950$; therefore his expectation $= \dfrac{1}{4950}(45 \times £100 + 570 \times £50 + 330 \times £20) = £8$.

6525. (By H. McColl, B.A.)—If n quantities be each taken at random between a and $-a$, show that the chance that their sum will be between b and $-b$ is

$$1 - \frac{2}{(2a)^n\, n!}\left\{(na-b)^n - n\,(na-b-2a)^n + \frac{n\,(n-1)}{1 \cdot 2}(na-b-4a)^n - \frac{n\,(n-1)\,(n-2)}{1 \cdot 2 \cdot 3}(na-b-6a)^n + \dots\right\} \text{ \&c.,}$$

the series to be continued as long as $na - b - 2ra$ is positive.

Solution by G. F. WALKER, M.A.; Rev. U. J. KNISELY, M.A.; *and others.*

The required chance $= \iiint \dots dx_1\, dx_2 \dots dx_n \div (2a)^n$,

where the integral is taken subject to the conditions

$$x_1 + x_2 + x_3 \dots + x_n > -b, \ < b,$$

and all the quantities between $-a$, $+a$. Add a to each; then, x_1', &c. being $= x_1 + a$, &c., $x_1' + x_2' \dots x_n' > na - b < na + b$, and x_1', &c. lie between 0 and $2a$, *i.e.*, are positive and $< 2a$. Now, the value of the integral

$$\iiint \dots dx_1'\, dx_2' \dots dx_n',$$

with the condition

$x_1' + x_2' + \dots x_n' < na + b$ is $\dfrac{[\Gamma(1)]^n}{\Gamma(n)} \displaystyle\int_0^{na+b} n^{s-1}\, dh$, and this $= \dfrac{(na+b)^n}{n\,!}$.

If one of the quantities, as x_r', be greater than $2a$, take $x_r' - 2a = y_r$. Then $x_1' + x_2' + \dots y_r + \dots x_n' < na + b - 2a$, and all the quantities are positive, the integral subject to this condition is $\dfrac{(na+b-2a)^n}{n!}$, y_r taking the place of x_r', and so on, if more of the quantities are $> 2a$. Hence the value subject to the restriction that all the quantities are $< 2a$ is

$$\frac{1}{n\,!}\left[(na+b)^n - n\,(na+b-2a)^n + \frac{n\,(n-1)}{1\,.\,2}\,(na+b-4a)^n - \dots\right] = \frac{(1)}{n\,!}.$$

The series, to continue so long as $na + b - 2ra$, is positive. Similarly, the value of the integral, subject to the restriction that $x_1 + x_2 + \dots x_n < na - b$, and all the quantities $< 2a$, is

$$\frac{1}{n\,!}\left[(na-b)^n - n\,(na-b-2a)^n + \frac{n\,(n-1)}{1\,.\,2}\,(na+b-4a)^n - \dots\right] = \frac{(2)}{n}.$$

The series, to go on so long as $na - b - 2ra$, is positive. Hence required chance is $\qquad [(1)-(2)] \div (2a)^n\, n\,!$

Now, considering the coefficient of x^n on the two sides of the expansion of $e^{+bx}\,(e^{ax} - e^{-ax})^n = e^{bx}\,(2ax + \dots)^n$, we see that $(1)-(2)$ may be replaced by $(2a)^n\, n\,! - 2\,(2)$, and the chance is $1 - \dfrac{2}{(2a)^n\, n\,!}\,(2)$.

6556. (By C. LEUDESDORF, M.A.)—If

$3nz^2 = mx^2 - 2lxy$, $\quad 3n^2z = l^2y + 2lmx$, $\quad n^3 = l^2m$, $\quad$ prove that $z^3 = x^2y$.

Solution by Prof. SCOTT, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

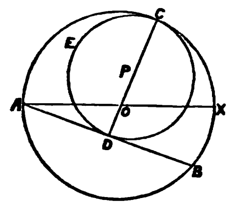
From $\quad 3nz^2 = mx^2 + 2lxy$, $\quad$ we get $3\,\dfrac{z_4}{n^2} = \dfrac{x^2}{l^2} + 2\,\dfrac{xy}{lm}$;

and from $\quad 3nz^2 = l^2y + 2lmx$, $\quad$ we get $3\,\dfrac{z}{n} = \dfrac{y}{m} + 2\,\dfrac{x}{l}$;

therefore $\qquad \dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$, and $z^3 = x^2y$.

6585. (By R. E. Riley, B.A.)—Any chord AB is drawn through A, a fixed point of a circle ABC, and a circle CDE is described touching ABC, and the chord AB at its middle point D; prove that the centre of CDE lies on a cardioid, whose pole is the centre of ABC.

Solution by R. Knowles, B.A., L.C.P.; J. O'Regan; *and others.*



Let $OP = r$, $\angle POX = \theta$, $AO = a$;

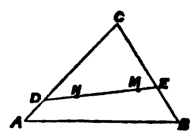
then $2OP = OC - OD$;

therefore $r = \frac{1}{2}a(1-\cos\theta)$,

which is the equation to a cardioid.

6027. (By Professor Seitz, M.A.)—Two points are taken at random in a triangle, and a line drawn through them. If two other points be taken at random in the triangle, show that the chance that they will both fall on the same side of the line, is $\frac{11}{18}$.

Solution by the Proposer.



Let ABC be the triangle, M and N two points such that DE, the line through them, meets AC and BC, and makes an angle with AC less than the angle A.

Let $CD = x$, $DM = y$, $MN = z$, $DE = y'$, $\angle CDE = \theta$, area $CDE = v$, $\dfrac{b^2 \sin C \sin\theta}{2\sin(\theta + C)} = u$, and area $ABC = \frac{1}{2}ab\sin C = 1$.

Then we have $y' = \dfrac{x\sin C}{\sin(\theta+C)}$, $v = \dfrac{x^2 \sin C \sin\theta}{2\sin(\theta+C)}$; an element of the triangle at M is $\sin\theta\, dx\, dy$, and at N it is $d\theta\, z\, dz$.

While θ varies from 0 to A, u varies from 0 to 1; and while x varies from 0 to b, v varies from 0 to u. The limits of y are 0 and y', and those of z are 0 and y, and doubled.

The number of ways two points can be taken in the triangle on the same side of DE is $v^2 + (1-v)^2$; hence, since the whole number of ways the four points can be taken is expressed by unity, we have

$$2\int_0^A \int_0^b \int_0^{y'} \int_0^y [v^2 + (1-v)^2] \sin\theta\, d\theta\, dx\, dy\, z\, dz$$

$$= \tfrac{1}{3}\int_0^A \int_0^b [v^2 + (1-v)^2] \sin\theta\, d\theta\, y'^3\, dx = \tfrac{2}{3}\int_0^1 \int_0^u [v^2 + (1-v)^2]\, u^{-1}\, du\, v\, dv$$

$$= \tfrac{1}{9}\int_0^1 (3 - 4u + 3u^2)\, u\, du = \tfrac{11}{108}.$$

This result is the same as that for each of five other expressions similar to that above. Hence the chance in question is $\frac{11}{18}$.

6394. (By W. J. C. SHARP, M.A.)—Find the area contained between an ordinary catenary, the tractrix which is its involute, and any tangent to the first.

Solution by D. EDWARDES; BELLE EASTON; *and others.*

Measuring s from the vertex, and ϕ from the tangent at the vertex, we have $s = c\tan\phi$; hence the required area is

$$\tfrac{1}{2}\int_0^{\phi} c^2\tan^2\phi\,d\phi = \tfrac{1}{2}c^2(\tan\phi - \phi).$$

[This is an example of the proposition that the area swept over by a cord unwound from the curve, whose intrinsic equation is $s = f(\phi)$, is $\int_{\phi_1}^{\phi_2}\{f(\phi)\}^2\,d\phi$. Similarly $\int_{\phi_1}^{\phi_2}\{f'(\phi)\}^2\,d\phi$ is the area swept over by the radius of curvature.]

5080. (By Professor SYLVESTER, F.R.S.)—If the coordinates (x, y, z) of a curve be proportional to three given cubic functions of t; prove the existence and find the positions of the node, in terms of the coefficients, and extend the method to the case of unicursal curves of any order.

Solution by W. J. C. SHARP, M.A.

If $px = a_0t^3 + a_1t^2 + a_2t + a_3$ and $py = b_0t^3 + b_1t^2 + b_2t + b_3$, the value of pz follows as a consequence of the linear identical relation between x, y, z.

At an ordinary point on the curve, the equations

$$a_0t^3 + a_1t^2 + a_2t + a_3 - px = 0 \text{ and } b_0t^3 + b_1t^2 + b_2t + b_3 - py = 0$$

must have a common root, the condition for which is the non-homogeneous equation to the curve, which is therefore a cubic.

Similarly, at a node, these equations must have two common roots, and the node will be a crunode, a cusp, or an acnode, according as these are real and unequal, equal or imaginary. Hence we have

$$(a_0t + \lambda)(b_0t^3 + b_1t^2 + b_2t + b_3 - py) \equiv (b_0t + \mu)(a_0t^3 + a_1t^2 + a_2t + a_3 - px),$$

therefore $$\left\|\begin{matrix} a_0, & a_1, & a_2, & a_3 - px \\ b_0, & b_1, & b_2, & b_3 - py \\ a_0b_1 - a_1b_0, & a_0b_2 - a_2b_0, & a_0(b_3 - py) - b_0(a_3 - px), & 0 \end{matrix}\right\| = 0;$$

equations which determine x and y uniquely; and consequently the curve has one and only one node, the coordinates of which are determined by the above equations.

In the general case, it is convenient to proceed somewhat differently.

If $\quad px = (a_0, a_1, a_2 \dots a_n)(t, 1)^n, \quad py = (b_0, b_1, b_2 \dots b_n)(t, 1)^n,$

then $$pz = (c_0, c_1, c_2 \dots c_n)(t, 1)^n,$$

where $\lambda a_0 + \mu b_0 + \nu c_0 =$ a constant dependent upon the system of coordi-

nates, and $\lambda a_r + \mu b_r + \nu c_r = 0$. The equation to the tangent at the point t is

$$\left|\begin{matrix} (a_1, a_2 \dots a_n) & (t, 1)^{n-1} \,.\, (b_1, b_2 \dots b_n) & (t, 1)^{n-1} \,.\, (c_1, c_2 \dots c_n) & (t, 1)^{n-1} \\ (a_0, a_1 \dots a_{n-1}) & (t, 1)^{n-1} \,.\, (b_0, b_1 \dots b_{n-1}) & (t, 1)^{n-1} \,.\, (c_0, c_1 \dots c_{n-1}) & (t, 1)^{n-1} \\ x & \quad . \quad y & \quad . \quad z & \end{matrix}\right| = 0,$$

so that the curve is of the $2(n-1)^{\text{th}}$ class at most; and, for a cusp,

$$[a_0, a_1, a_2 \dots (a_n - px)]\,(t, 1)^n = 0 \text{ and } [b_0, b_1, b_2 \dots (b_n - py)]\,(t, 1)^n = 0 \;\dots\text{(A)}$$

must have a square common factor $(t-\alpha)^2$, and $t-\alpha$ will divide the second line of the determinant, so that the class is reduced by one for each cusp, and is $2(n-1)-k$, and the order is n; therefore $2(n-1)-k = n^2-n-2\delta-3k$, and $k+\delta = \frac{1}{2}(n-1)(n-2)$, *i.e.* the deficiency is zero; also if $U = 0$ be the resultant of the equations (A), at a node where they have two common roots $\frac{dU}{dx} = 0$ and $\frac{dU}{dy} = 0$, which equations will determine the nodes,

and the common roots satisfy $\;t^2 \frac{d^2U}{dx^2} - 2\frac{t}{n}\frac{d^2U}{dx\,da_{n-1}} + \frac{1}{n^2}\frac{d^2U}{da_{n-1}^2} = 0,$

so that any node will be a crunode, a cusp, or an acnode, according as

$$\left(\frac{d^2U}{dx\,da_{n-1}}\right)^2 > = < \frac{d^2U}{dx^2}\,\frac{d^2U}{da_{n-1}^2}.$$

Similarly, if $(a_0 \,.\, a_1 \,.\, a_2 \dots a_n)\,(x \,.\, 1) = 0$ and $(b_0 \,.\, b_1 \,.\, b_2 \dots b_n)\,(x \,.\, 1) = 0$ have a pair of common roots, these will be unequal and real, equal or imaginary, according as $\left(\frac{d^2k}{da_n \,.\, da_{n-1}}\right)^2 > = < \frac{d^2k}{da_n^2} \,.\, \frac{d^2k}{da_{n-1}^2}.$

[Another solution may be seen in *Reprint*, Vol. XXVI., pp. 83—86.]

6608. (By the Editor.)—Find, to four places of decimals, and, if possible, in a finite form, the value of the following series and its equivalent integral, deduced in Professor Seitz's solution of the Editor's Question 6482 (*Reprint*, Vol. XXXV., p. 30) :—

$$\frac{1}{2^2} - (1-\tfrac{1}{2})\frac{1}{3^2} + (1-\tfrac{1}{2}+\tfrac{1}{3})\frac{1}{4^2} - (1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4})\frac{1}{5^2} + \dots \equiv \int_0^1 \frac{\log(1-x)}{1+x}\log x\,dx.$$

Solution by W. H. Blythe, B.A.

1. The series $\quad \frac{1}{n^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} - \frac{1}{(n+3)^2} + \dots$

lies between $\quad \frac{1}{n(n-1)(n+1)} + \frac{1}{n(n+1)(n+2)} + \dots = \frac{1}{2n(n-1)}$

and $\quad \frac{1}{n(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots = \frac{1}{2(n+1)n}.$

If n be less than 25, both these expressions agree to four places of decimals,

$$\frac{1}{25^2} - \frac{1}{26^2} + \dots = \cdot 0008 \text{ nearly}, \quad \frac{1}{26^2} - \frac{1}{27^2} + \dots = \cdot 00074 \text{ nearly}.$$

Also, since the general term of the given series is

$$\frac{1}{n}\left(\frac{1}{(n+1)^2}-\frac{1}{(n+2)^2}+\dots\text{ to infinity}\right),$$

taking this as lying between $\dfrac{1}{2n^2(n+1)}$ and $\dfrac{1}{2n(n+1)(n+2)}$, we get *a fortiori* between $\dfrac{1}{2(n+1)(n-1)n}$ and $\dfrac{1}{2n(n+1)(n+2)}$, and hence the sum of all remaining terms after the nth lie between $\dfrac{1}{4(n-1)n}$ and $\dfrac{1}{4n(n+1)}$. We find that, after the sum of 21 terms, these two must agree to four places of decimals, and $=\cdot0006$ nearly. Calculating the first 20 terms, using the above approximation for squares after 25^{-2}, and putting $\cdot0006$ for the whole remainder of the series, the sum to four places of decimals is found to be $\cdot242894$. In fact, the series is as follows:—

·178753	·000841	·000138
·034921	·000610	·000119
·012880	·000438	·000095
·005933	·000341	·000079
·003228	·000255	·000069
·001913	·000210	·000058
·001253	·000160	·000600

I have arranged the series thus:—

$$\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\dots\right)+\frac{1}{2}\left(\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\dots\right)+\dots\frac{1}{n}\left(\frac{1}{(n+1)^2}+\dots\right).$$

2. *Otherwise*:—The series

$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\right)\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\dots\right)$$

$$+\frac{1}{2}\cdot\frac{1}{2^2}-\frac{1}{3}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\frac{1}{4^2}\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}\right)$$

$$-\frac{1}{5}\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}\right)+\frac{1}{6}\left(\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{6^2}\right)-\dots$$

$$=\log_e 2\left(1-\tfrac{1}{12}\pi^2\right)+\dots=(\cdot6931)(\cdot1775)+\dots$$

$$=\cdot1231+\cdot125-\cdot04\dot{6}6+\cdot0506-\dots$$

It will be found that, after a certain number of terms, each is less than the preceding and of opposite sign; the series is convergent, and will be found to agree with the past result, that is, $\cdot2429$ nearly.

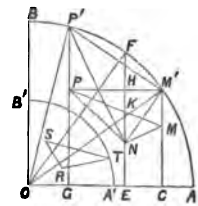
6285 & 6477. (By Professor MATZ, M.A.)—6285. Three points being taken at random in the surface of a circular quadrant; prove that the mean value of all the triangles that can be formed by joining the three points is

$$\frac{r^2}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right).$$

6477. **Two points are (1) taken at random on the *arc*, and another point in the *surface* of a circular quadrant; and again, two points are (2) taken at random in the *surface*, and another point on the *arc* of a circular quadrant; prove that the average areas of the triangles formed by joining these points in their respective order will be to each other as**

$$45\pi^2+84\pi-684 : 35\pi^2+64\pi-524.$$

Solution by Professor Seitz, M.A.



(6285.) Let AOB be the quadrant, and M, N, P the random points in its surface. Through M, N, P draw CM′, EF, GP′ perpendicular to AO, EF intersecting MP in K. Now it is necessary to consider only those relative positions of the points in which CM′ lies to the right of GP′, and EF between CM′ and GP′; since under these limitations the triangle will pass through all the possible variations.

Let $OA=r$, $GP=x$, $CM=y$, $EN=z$, $GP'=x'$, $CM'=y'$, $EF=z'$, $EK=w$, $\angle GOP'=\theta$, $\angle COM'=\phi$, $\angle EOF=\psi$, and $v=1+(\cos\phi-\cos\theta)$. Then we have $x'=r\sin\theta$, $y'=r\sin\phi$, $z'=r\sin\psi$, $w=xv(\cos\phi-\cos\psi)+yv(\cos\psi-\cos\theta)$; the area of the triangle MNP is $u=\dfrac{r}{2v}(w-z)$, when $z<w$; and the area is $u_1=\dfrac{r}{2v}(z-w)$, when $z>w$.

An element of surface at M is $r\sin\phi\,d\phi\,dy$, at N it is $r\sin\psi\,d\psi\,dz$, and at P it is $r\sin\theta\,d\theta\,dx$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; of ψ, ϕ and θ; of x, 0 and x'; of y, 0 and y'; of z, 0 and w, and w and z'.

Hence, since the whole number of ways the three points are taken is $\frac{1}{6}(\frac{1}{4}\pi r^2)^3$, the average area of the triangle is

$$\Delta=\frac{384}{\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\left\{\int_0^{w}u\,dz+\int_{w}^{z'}u_1\,dz\right\}r\sin\theta\,d\theta\,r\sin\phi\,d\phi\,r\sin\psi\,d\psi\,dx\,dy$$

$$=\frac{32r^2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\left\{\begin{array}{l}2\sin^3\theta\sin\phi(\cos\phi-\cos\psi)^2+2\sin\theta\sin^3\phi(\cos\psi-\cos\theta)^2\\ +3\sin^2\theta\sin^2\phi(\cos\phi-\cos\psi)(\cos\psi-\cos\theta)\\ +3\sin\theta\sin\phi\sin^2\psi(\cos\phi-\cos\theta)^2\\ -3\sin^2\theta\sin\phi\sin\psi(\cos\phi-\cos\theta)(\cos\phi-\cos\psi)\\ -3\sin\theta\sin^2\phi\sin\psi(\cos\phi-\cos\theta)(\cos\psi-\cos\theta)\end{array}\right\}\times\sin\theta\sin\phi\sin\psi\,v\,d\theta\,d\phi\,d\psi$$

$$=\frac{16r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\begin{array}{l}4\sin^2\theta\cos^2\phi+4\sin^2\phi\cos^2\phi+4\sin^2\theta\cos^2\theta\\ +4\sin^2\phi\cos^2\theta-5\sin^2\theta\cos\theta\cos\phi-5\sin^2\phi\cos\theta\cos\phi\\ -6\sin\theta\cos\theta\sin\phi\cos\phi+12+6\cos^2\theta+6\cos^2\phi\\ -24\cos\theta\cos\phi-12\sin\theta\sin\phi-9(\theta-\phi)\sin(\theta-\phi)\end{array}\right\}\times\sin^2\theta\sin^2\phi\,d\theta\,d\phi$$

$$=\frac{8r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta\,d\theta$$

$$=\frac{r^2}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right).$$

(6477.) (1) Let M′, P′ be two random points in the arc, M′ being nearest A, and N a random point in the surface of the quadrant.

Let $OA = r$, $EN = x$, $EF = x'$, $EH = y$, $\angle AOP' = \theta$, $\angle AOM' = \phi$, $\angle AOF = \psi$, and $v = 1 \div (\cos\phi - \cos\theta)$. Then

$$x' = r\sin\psi,\quad y = rv\sin\theta(\cos\phi - \cos\psi) + rv\sin\phi(\cos\psi - \cos\theta);$$

if N is in the segment ACM′, the trapezoid CGP′M′, or the segment GOBP′, the area of the triangle M′NP′ is $u = \dfrac{r}{2v}(y-x)$; and if N is in the segment M′FP′, the area of the triangle is $u_1 = \dfrac{r}{2v}(x-y)$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and θ; from $\psi = 0$ to $\psi = \phi$ the limits of x are 0 and x', from $\psi = \phi$ to $\psi = \theta$ they are 0 and y, and y and x', and from $\psi = \theta$ to $\psi = \frac{1}{2}\pi$ they are 0 and x'.

Hence, since the whole number of ways the three points are taken is $\frac{1}{2}(\frac{1}{4}\pi r^2)(\frac{1}{2}\pi r)^2$, the average area of the triangle is

$$\Delta_1 = \frac{32}{\pi^3 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{\int_0^{\phi}\int_0^{x'} u\,r\sin\psi\,d\psi\,dx + \int_{\phi}^{\theta}\left(\int_0^{y} u\,dx + \int_y^{x'} u_1\,dx\right) r\sin\psi\,d\psi\right.$$
$$\left. + \int_{\theta}^{\frac{1}{2}\pi}\int_0^{x'} u\,r\sin\psi\,d\psi\,dx\right\} r\,d\theta\,r\,d\phi$$

$$= \frac{8r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left\{3(\tfrac{1}{2}\pi - 2\theta + 2\phi)\sin(\theta-\phi) - 2\sin\theta + 2\sin\phi + 2\cos\theta\right.$$
$$\left. - 2\cos\phi + 8 - 8\cos(\theta-\phi) + 2\sin^2(\theta-\phi)\right\} d\theta\,d\phi$$

$$= \frac{4r^2}{3\pi^3}\int_0^{\frac{1}{2}\pi}(3\pi - 3\pi\cos\theta + 18\theta + 16\theta\cos\theta - 4\theta\sin\theta + 4 - 32\sin\theta - 4\cos\theta$$
$$- 2\sin\theta\cos\theta)\,d\theta$$

$$= \frac{8r^2}{\pi}\left(\frac{5}{8} + \frac{7}{6\pi} - \frac{19}{2\pi^2}\right).$$

(2) Let N, P be two random points in the surface, and M′ a random point in the arc of the quadrant. Draw the quadrant A′OB′.

Let two points, R, S, be taken at random in the surface of this quadrant, and a third point, T, in its arc.

Let $OA' = x$, and Δ = the average area of the triangle M′NP.

Then for a given value of x the average area of the triangle $RST = x^2\Delta_, \div r^2$, and the average area of the triangle MNP is

$$\Delta = \int_0^r \frac{x^2}{r^2}\Delta_2 \cdot \tfrac{1}{2}\pi x\,(\tfrac{1}{4}\pi x^2)^2\,dx \div \int_0^r \tfrac{1}{2}\pi x\,(\tfrac{1}{4}\pi x^2)^2\,dx = \tfrac{3}{4}\Delta_,.$$

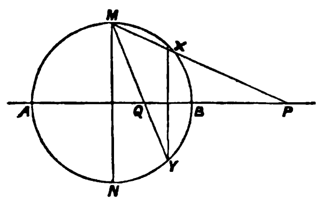
Hence we have $\quad \Delta_2 = \frac{4}{3}\Delta = \dfrac{r^2}{\pi}\left(\dfrac{35}{9} + \dfrac{64}{9\pi} - \dfrac{524}{9\pi^2}\right),$

and $\quad \Delta_1 : \Delta_2 = 45\pi^2 + 84\pi - 684 : 35\pi^2 + 64\pi - 524.$

6547. (By E. B. Elliott, M.A.)—From a theorem in plane geometry another is obtained by inversion with regard to any circle in the plane; and from the two theorems, by stereographic projection on a sphere having

for centre and radius the centre and radius of inversion, two theorems of spherical geometry are obtained. Prove that these two are identical.

Solution by Professor TOWNSEND, M.A., F.R.S.



Denoting by M and N the two extremities of the diameter of the sphere perpendicular to the plane of the circle, by P and Q any pair of inverse points on any diameter AB of the circle, and by X and Y the stereographic projections of P and Q from either M or N on the surface of the sphere; then, since, on account of the harmonic section of AB by P and Q, the chord XY is always parallel to the diameter MN of the sphere, all pairs of curves traced out by X and Y on the sphere are reflections of each other with respect to the plane of the circle; therefore, &c.

6162. (By Professor TANNER, M.A.)—Required the form of $\phi(x)$ such that $\dfrac{x\phi'(y)-y\phi'(x)}{x^2\phi(y)-y^2\phi(x)}$ may be a function of xy.

Solution by J. HAMMOND, M.A.; CHRISTINE LADD; *and others.*

$\dfrac{x\phi'(y)-y\phi'(x)}{x^2\phi(y)-y^2\phi(x)}$ = Funct. (xy) is evidently satisfied by $\phi(x)=x^2$. Assume $\phi(x)=x^2\psi(x)$ for the general solution; then we have

$$\frac{2xy\{\psi(y)-\psi(x)\}+xy\{y\psi'(y)-x\psi'(x)\}}{x^2y^2\{\psi(y)-\psi(x)\}}=\text{Funct. }(xy);$$

or
$$\frac{y\psi'(y)-x\psi'(x)}{\psi(y)-\psi(x)}=\text{Funct. }(xy).$$

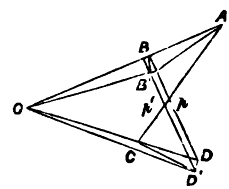
Putting $x=0$, $\dfrac{y\psi'(y)}{\psi(y)-a}=b$, a and b being arbitrary constants; solving this equation, $\psi(y)-a=cy^b$, and $\phi(x)=x^2\psi(x)=ax^2+cx^{b+2}$.

[The PROPOSER remarks that this solution is not, he thinks, complete. The form $\psi x=a+bx^n+cx^{-n}\ldots(1)$ also satisfies the first equation for ψ; a, b, c, n being arbitrary. Moreover, corresponding to the limiting case of $n=0$, we have a form which, after some reductions, may be written $\psi x=a+b\log x+c(\log x)^2\ldots(2)$, where a, b, c are arbitrary. The property of ϕx is the necessary and sufficient condition that the equation $\dfrac{dx}{(\phi x)^{\frac12}}+\dfrac{dy}{(\phi y)^{\frac12}}\ldots(3)$ should have an integral $x(\phi y)^{\frac12}+y(\phi x)^{\frac12}=\chi(xy)$. In

fact, the expression given in the question is $= 2\chi' : \chi$. The equation (3) includes as a particular case the well-known equation of the elliptic addition-theorem: viz., putting $n = 2$ in (1), we have $\phi x, = x^2\psi x, = c + ax^2 + bx^4$. The method of integration is STURM's.]

6605. (By Professor PURSER, M.A.)—Four bars are jointed together at the ends, two of them crossing each other; show that, when the four ends lie on the same circle, the difference of the areas of the triangles formed by the bars is less than in any other position.

Solution by G. F. WALKER, M.A.; A. ANDERSON, B.A.; *and others.*



Let AB, BD, DC, CA be the four rods, AC and BD being crossed at p, and let AB, DC meet in O; keep AC fixed, and let BD be displaced to B′D′. This will be about O as instantaneous centre, and BOB′ = DOD′. The difference $ABp \sim CpD$ will be the same as the difference $AB'p' \sim Cp'D'$ provided the small triangles AB′O, CD′O be equal; that is, if

$$OA \cdot BB' = OC \cdot DD'.$$

But $BB' = OB \cdot \theta$, and $DD' = OD \cdot \theta$, if θ be the small angle turned through; therefore $\quad OA \cdot OB = OC \cdot OD,$

or the four points A, B, C, D lie on a circle.

[The PROPOSER remarks that this no doubt proves that the difference of the areas is stationary when the four points lie on a circle, but that it would appear to require further analysis to show that the *stationary* answered to a minimum position.]

6506. (By Professor CROFTON, F.R.S.)—Express the sum of the series (n being integral) $\quad S = x^n + n^2x^{n-1} + \dfrac{n^2(n-1)^2}{1 \cdot 2}x^{n-2} + \&c.$

Solution by G. HEPPEL, M.A.; Prof. PRATT, M.A.; *and others.*

To actually *sum* the series is a difficult business, though it is easy enough to *express the sum*, for

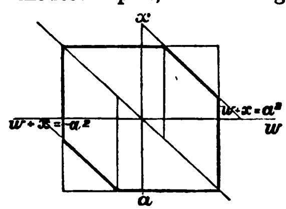
$$S = \frac{1}{e^x}\frac{d^n(e^x \cdot x^n)}{dx^n}.$$

NOTE ON QUESTION 6545. *By* ELIZABETH BLACKWOOD.

The Solution of this Question, given on p. 61 of this volume, is certainly wrong *on the supposition* $a<2$. The limits of y in the first negative integral imply that $a>(w+x)a^{-1}$, which implies that $x<a^2-w$. If the limit a of the variable x supersedes the limit a^2-w (as the integral implies), the former limit must be nearer to x, so that we have $a<a^2-w$, and therefore $w<a^2-a$. If the limit a of the variable w supersedes a^2-a (as the integral also implies), we must have $a<a^2-a$, that is, $a>2$, *which is contrary to the hypothesis.* In the same way the next three integrals may be shewn to be inconsistent with the supposition $a<2$. I may further remark that the solution is *geometrical* as regards the limits of y and z only, being a mere *ipse dixit* as regards the limits of w, x, and the constant a.

NOTE ON QUESTION 6545. *By* G. F. WALKER, M.A.

On referring to my solution, I see that, in the second part, the following correction is required:—



If $a<2$, the hyperbola does not always meet the square, and the point w, x must be confined to the figure bounded by the black line, and *then* this integral *added*:—

$$\int_{a^2-a}^{a}\int_{a^2-w}^{a}\int_{-a}^{a}\int_{-a}^{a} \mathrm{P}\,dx\,dy\,dz.$$

Each of the integrals with respect to x, w splits up into two, making four, taken each over one of the compartments into which the above area is divided.

6639. (By J. J. WALKER, M.A.) — The constants a, b, c of an empirical formula $p=a+b\tan(\theta+c)$ are to be determined from three pairs of observed simultaneous values of the variables, viz., $p\theta$, $p'\theta'$, $p''\theta''$; prove that (1) the single real value of c is that given by

$$\tan c=\Sigma\,p\cos\theta\sin(\theta'-\theta'')\div\Sigma\,p\sin\theta\sin(\theta'-\theta'');$$

and (2), if $\quad (p'-p'')^2\cos(\theta'-\theta'')\sin(\theta''-\theta)\sin(\theta-\theta')=q\,\ldots\ldots,$

$$a=(pq+p'q'+p''q'')\div(q+q'+q''),$$

$$b=-(p'-p'')(p''-p)(p-p')\sin(\theta'-\theta'')\sin(\theta''-\theta)\sin(\theta-\theta')\div(q+q'+q'').$$

Solution by G. HEPPEL, M.A.; BELLE EASTON; *and others.*

1. From the three original equations we obtain

$$\frac{p-p'}{p'-p''}=\frac{(\tan\theta-\tan\theta')(1-\tan\theta''\tan c)}{(\tan\theta'-\tan\theta'')(1-\tan\theta\tan c)};$$

$$\therefore\quad \tan c=\frac{\Sigma\,p(\tan\theta'-\tan\theta'')}{\Sigma\,p\tan\theta(\tan\theta'-\tan\theta'')}=\frac{\Sigma\,p\cos\theta\sin(\theta'-\theta'')}{\Sigma\,p\sin\theta\sin(\theta'-\theta'')}.$$

2. Let $p'-p''=r$, $\sin(\theta'-\theta'')=u$, $\cos(\theta'-\theta'')=v$; then

$$\tan(\theta+c)=\frac{\sin\theta+\cos\theta\tan c}{\cos\theta-\sin\theta\tan c}=\frac{pu+p'u'v''+p''u''v'}{(p''-p')\,u'u''};$$

and, since $u=-u'v''-u''v'$, we get

$$\tan(\theta+c)=\frac{r''u'v''-r'u''v'}{ru'u''};\ \text{ similarly }\ \tan(\theta'+c)=\frac{ru''v-r''uv''}{r'u''u};$$

hence $$\frac{p-a}{p'-a}=\frac{\tan(\theta+c)}{\tan(\theta'+c)}=\frac{rr''uu'v''-r'^2u''uv'}{r^2u'u''v-rr''uu'v''}.$$

Noticing that $r'+r''=-r$, and that $p''r'+p'r''=-pr$, the above equation at once reduces to

$$a[q+q'+q'']=pq+p'q'+p''q'',\quad p-p'=b\left[\frac{r''u'v''-r'u''v'}{ru'u''}-\frac{ru''v-r''uv''}{r'u''u}\right];$$

therefore $$rr'r''uu'u''=-b(q+q'+q'').$$

6561. (By E. W. Symons, M.A.)—A conic is self-conjugate to the triangle of reference; (x', y', z') are the areal coordinates of its centre, and (x, y, z) the distances of this point from the vertices of the triangle; prove that the radius of the director circle is $(x'x^2+y'y^2+z'z^2)^{\frac{1}{2}}$.

Solution by W. H. Blythe, B.A.; Belle Easton; *and others.*

The equation to the conic is $\dfrac{a^2}{x_1}+\dfrac{\beta^2}{y_1}+\dfrac{\gamma^2}{z_1}=0$; and if this be transformed to rectangular axes through the centre to $Ax^2+2Hxy+By^2=1$,

we have $$A+B\equiv\left(\frac{a^2}{x_1}+\frac{b^2}{y_1}+\frac{c^2}{z_1}\right)+4\Delta^2,$$

$$AB-H^2\equiv\frac{1}{4\Delta^2}\left(\frac{a^2b^2\sin^2C}{x_1y_1}+\frac{b^2c^2\sin^2A}{y_1z_1}+\frac{c^2a^2\sin^2B}{z_1x_1}\right)\equiv\frac{1}{x_1y_1z_1\cdot4\Delta^2}.$$

The square on the radius of the director circle equals sum of squares on semi-axes $=\dfrac{A+B}{AB-H^2}=a^2y_1z_1+b^2x_1z_1+c^2x_1y_1$. If a, β, γ; a_1, β_1, γ_1, be two points, the distance between them is the square root of

$$\frac{b^2+c^2-a^2}{2}(a-a_1)^2+\frac{c^2+a^2-b^2}{2}(\beta-\beta_1)^2+\frac{a^2+b^2-c^2}{2}(\gamma-\gamma_1)^2.$$

Hence, if x be the distance between (x_1, y_1, z_1) and $(1, 0, 0)$,

$$x^2=\frac{b^2+c^2-a^2}{2}(x_1-1)^2+\frac{c^2+a^2-b^2}{2}y_1^2+\frac{a^2+b^2-c^2}{2}z_1^2,$$

with similar expressions for y^2 and z^2. Multiply x^2 by x_1, y^2 by y_1, z^2 by

z_1, and, remembering that $x_1+y_1+z_1 = 1$, we get

$$x_1x^2+y_1y^2+z_1z^2 = \frac{b^2+c^2-a^2}{2}(y_1x_1+z_1x_1)+\text{two similar terms}$$

$$= a^2y_1z_1+b^2x_1z_1+c^2x_1y_1,$$

which has been shown $= r^2$, where r = radius of director circle;

hence we have $$r = (x_1x^2+y_1y^2+z_1z^2)^2.$$

6638. (By H. G. Dawson, M.A.)—If four of the roots of

$$(a,\ b,\ c,\ d,\ e,\ f)(x,\ 1)^5 = 0$$

be connected by the relation $\alpha+\beta = \gamma+\delta$, prove that the remaining root ϵ is given by the equations $z = a\epsilon+b$, $z^3-8Hz+16G = 0$, where $H = b^2-ac$, $G = 2b^3-3abc+a^2d$.

Solution by the Rev. T. R. Terry, F.R.A.S.; J. O'Regan; *and others.*

Multiplying throughout by a^4, and putting $y = ax+b$, we get

$$y^5-10Hy^3+10Gy^2+\ldots = 0.$$

One root of this equation is z, and the others are connected by the relation

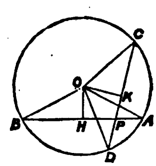
$$\alpha+\beta = \gamma+\delta, \quad \text{therefore} \quad \alpha+\beta = \gamma+\delta = -\tfrac{1}{2}z,$$

$$-10H = \alpha\beta+\gamma\delta-\tfrac{3}{4}z^2, \quad -10G = \tfrac{1}{2}z(\alpha\beta+\gamma\delta)+\tfrac{1}{4}z^3.$$

Hence $$Hz+16G = 0.$$

6314. (By Professor Seitz, M.A.)—In the surface of a circle two lines are drawn at random in length and direction; prove that the chance that they intersect is $\frac{3}{32}$.

Solution by the Proposer; Prof. Matz, M.A.; *and others.*



Let AB and CD be two chords intersecting in P, and O the centre of the circle. Draw OH and OK perpendicular to AB and CD.

Let $OA = r$, $PA = x$, $PB = y$, $\angle AOH = \theta$, $\angle HOK = \phi$, and $\angle COK = \psi$. Then $AB = 2r\sin\theta$, $CD = 2r\sin\psi$, and

$$xy = r^2[\sin^2\theta-(\cos\psi\operatorname{cosec}\phi-\cos\theta\cot\phi)^2].$$

The number of ways two lines can be drawn, one in AB and the other in CD, so as to intersect, is $4x^2y^2$, and the whole number of ways they can be drawn in these chords is $AB^2 \cdot CD^2 = 16r^4\sin^2\theta\sin^2\psi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$; and for intersection the limits of ψ must be $\pm(\theta-\phi)$ and $\theta+\phi$ (the double sign being taken positive when $\phi < \theta$, and negative when $\phi > \theta$).

Hence the required chance is

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} 4x^2y^2\, r\sin\theta\, d\theta\, d\phi\, r\sin\psi\, d\psi + \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi} 16r^6\sin^3\theta\sin^3\psi\, d\theta\, d\psi$$

$$= \frac{9}{32\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} [\sin^2\theta - (\cos\psi\,\mathrm{cosec}\,\phi - \cos\theta\cot\phi)^2]^2 \sin\theta\, d\theta\, d\phi\sin\psi\, d\psi$$

$$= \frac{3}{10\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^6\theta\sin\phi\, d\theta\, d\phi = \frac{3}{10\pi}\int_0^{\frac{1}{2}\pi}\sin^6\theta\, d\theta = \frac{3}{32}.$$

6656. (By W. H. Besant, F.R.S.) — Find (1) the volume; (2) the surface generated by the revolution about the initial line of the curve $r = a(1+\cos\theta)$; also (3) the mean value of the radii to all points within the volume enclosed; and prove (4) that the mean value of the radii to the surface is equal to a when the radii are drawn symmetrically in space, and to $\frac{10}{7}a$ when the surface is covered equably by the ends of the radii; also (5) that, if the volume be made up of a series of shells of equal volumes, bounded by a succession of surfaces similar and similarly situated to the outer surface, and having the same cuspidal point, the mean value of their surfaces is three-fifths of the outer surface.

Solution by the Rev. T. R. Terry, F.R.A.S.; Prof. Droz; *and others.*

Let M_1, M_2, M_3, M_4 be the four mean values to be found; then

(1) $$V = \text{volume} = \int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r\sin\theta\,.\, r\, d\theta\, dr = \frac{8\pi a^3}{3}.$$

(2) Since element of arc $= 2a\cos\frac{1}{2}\theta\, d\theta$, therefore

$$S = \text{surface} = \int_0^{\pi} 2\pi r\sin\theta\,.\,2a\cos\frac{\theta}{2}\, d\theta = \frac{32\pi a^2}{5}.$$

(3) $$M_1 V = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,.\,2\pi r\sin\theta\,.\,r\, d\theta\, dr = \frac{16\pi a^4}{5}, \quad \therefore\ M_1 = \frac{6a}{5}.$$

(4) $$4\pi M_2 = \int_0^{\pi} r\,.\,2\pi\sin\theta\, d\theta = 4\pi a, \quad \text{therefore } M_2 = a.$$

$$M_3 S = \int_0^{\pi} r\,.\,2\pi r\sin\theta\,.\,2a\cos\frac{\theta}{2}\, d\theta = \frac{64\pi a^3}{7}, \quad \text{therefore } M_3 = \frac{10a}{7}.$$

(5) Since $S = \lambda V^{\frac{2}{3}}$, where λ is a constant, we have

$$M_4 V = \int_0^{V} \lambda V^{\frac{2}{3}}\,.\,dV = \tfrac{3}{5}\lambda V^{\frac{5}{3}}, \quad \text{therefore } M_4 = \tfrac{3}{5}S.$$

6646. (By W. B. Grove, B.A.)—There are 24 cards, every one of which is painted with one or more of the three colours—red, blue, and yellow. Observation informs us that there are 15 painted with blue, 19

with yellow, and 16 not painted with red; 8 painted with both red and blue, 3 with both red and yellow, 10 with both blue and yellow. Find the law or laws which regulate the combinations of the colours.

Solution by ALEXANDER MACFARLANE, D.Sc., F.R.S.E.

Let U denote the 24 cards, r the portion which are painted red, b the portion which are painted blue, y the portion which are painted yellow. Then the data are

$$U = 24,\quad b = \frac{15}{24},\quad y = \frac{19}{24},\quad 1-r = \frac{16}{24},\quad rb = \frac{8}{24} \ldots (1, 2, 3, 4, 5),$$

$$ry = \frac{3}{24},\quad by = \frac{10}{24},\quad (1-r)(1-b)(1-y) = 0 \ldots\ldots (6, 7, 8).$$

From (5) and (4), we have $\quad rb = \frac{8}{24} = r \ldots\ldots (9).$

From (3), (6), (4), $\quad y(1-r) = y - yr = \frac{19}{24} - \frac{3}{24} = \frac{16}{24} = 1-r \ldots\ldots (10).$

From (2), (7), (3), $\quad b(1-y) = b - by = \frac{15}{24} - \frac{10}{24} = \frac{5}{24} = 1-y \ldots\ldots (11),$

But (11) follows from (9) and (10);

for $\quad r = \frac{0}{1-b}$ from (9), and $r = \frac{1-y}{1-y}$ from (10),

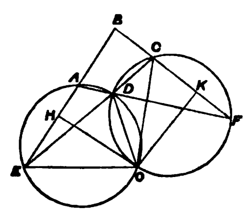
therefore $\quad \frac{0}{1-b} = \frac{1-y}{1-y},\quad$ therefore $0 = (1-b)(1-y)$,

which is equivalent to (11).

Hence the laws are as follows:—A card which is red is also blue; and a card which is not red, is yellow.

6641. (By W. S. McCAY, M.A.)—Prove that the quadrilinear coordinates of the focus of the parabola touching four lines x_1, x_2, x_3, x_4 are given by the equations $R_1x_1 = R_2x_2 = R_3x_3 = R_4x_4$, where R_1 is the radius of the circle circumscribing the triangle $x_2x_3x_4$, &c.

I. *Solution by* W. M. COATES, B.A.; LIZZIE A. KITTUDGE; *and others.*



Let ABCD be the quadrilateral formed by the four lines x_1, x_2, x_3, x_4; and O the focus of the parabola touching them.

Let fall perpendiculars OH and OK from O on AB and BC. Join OE, OC, and OD.

Now, since the circles circumscribing the four triangles formed by x_1, x_2, x_3, x_4 pass through O, the triangles OHE and OCK are equiangular, therefore

$$OH : OK = OE : OC = R_2 : R_1;$$

therefore $\quad x_1 : x_2 = R_2 : R_1,\;$ or $\; x_1R_1 = x_2R_2.$

Similarly for the others; therefore x_1, x_2, x_3, x_4 are given by

$$x_1R_1 = x_2R_2 = x_3R_3 = x_4R_4.$$

II. *Solution by the* PROPOSER.

The theorem will be proved if I show that the parameter of a parabola, which is inscribed in a triangle, and the coordinates of whose focus are (α, β, γ), is given by the equation $\frac{1}{m^2} = \frac{2R}{\alpha\beta\gamma}$, as most readily appears thus. For any point we have

$$\alpha \sin A + \beta \sin B + \gamma \sin C = \Delta R^{-1},$$

where Δ is the area of the triangle, and R the radius of the circumscribed circle.

Reciprocate the triangle to a circle of unit radius whose centre is $\alpha\beta\gamma$; the distances of the origin to the new vertices will be connected with the old triangle by the relation

$$\frac{\sin A}{r_1} + \frac{\sin B}{r_2} + \frac{\sin C}{r_3} = \frac{\Delta}{R},$$

and A, B, C being equal (because $\alpha r_1 = \beta r_2 = \gamma r_3$), or supplemental to the angles between r_1, r_2, r_0, this becomes

$$2\Delta' = \frac{\Delta}{R} \cdot \frac{1}{\alpha\beta\gamma},$$

where Δ' is the area of the new triangle. (This also follows from TOWNSEND's *Modern Geometry*, Vol. I., p. 234.)

If the origin be on the circumscribed circle of the old triangle, the two triangles are similar, and

$$\frac{\Delta'}{\Delta} = \frac{R'^2}{R^2}, \quad \text{hence} \quad R'^2 = \frac{R}{2\alpha\beta\gamma},$$

and the circumscribed circle of the new triangle being the reciprocal of the parabola in question, $2R'm = 1$, giving the required value of m.

The relations in the question may also be obtained by elementary geometry; they were first arrived at by elimination.

6615. (By E. B. ELLIOTT, M.A.)—Prove that

$$\int_0^a x^n (2a-x)^n \, dx = 2^{2n} \int_0^a x^n (a-x)^n \, dx \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

$$\frac{2^{2n}-1}{\Gamma(2n+2)} = \frac{1}{\Gamma(2n+1)} + \frac{1}{1.2}\cdot\frac{1}{\Gamma(2n)} + \frac{1}{1.2.3}\cdot\frac{1}{\Gamma(2n-1)} \;\ldots\ldots\ldots (2);$$

the series stopping at the n^{th} term if n be a positive integer or zero.

Solution by R. KNOWLES, B.A., L.C.P.; D. EDWARDES; *and others.*

1. The right-hand expression

$$= 2^{2n} \cdot a^{2n+1} \cdot \frac{\Gamma(n+1)\,\Gamma(n+1)}{\Gamma(2n+2)} = 2^{2n} \cdot a^{2n+1} \, \frac{n!\; n!}{(2n+1)!}.$$

Putting $x = a(1-\sin\theta)$, the left-hand expression

$$\equiv a^{2n+1}\int_0^{\frac{1}{2}\pi}\cos^{2n+1}\theta\,d\theta = \frac{2n(2n-2)\dots 2}{(2n+1)(2n-1)\dots 3}a^{2n+1} = 2^{2n}a^{2n+1}\cdot\frac{n!\,n!}{(2n+1)!}.$$

2. Since the expansion of $(1+1)^{2n+1}$ contains $2n+2$ terms, we have

$$2^{2n}-1 = \tfrac{1}{2}(1+1)^{2n+1}-1 = (2n+1)+\frac{(2n+1)(2n)}{1.2}+\dots \text{ to } n \text{ terms};$$

$$\therefore\quad \frac{2^{2n}-1}{\Gamma(2n+2)} = \frac{1}{\Gamma(2n+1)}+\frac{1}{1.2}\,\frac{1}{\Gamma(2n)}+\frac{1}{1.2.3}\,\frac{1}{\Gamma(2n-1)}+\dots\dots$$

6202. (By the Rev. W. A. WHITWORTH, M.A.)—Give, in as simple a form as possible, the arithmetical rule for finding the seventh root of a given number.

I. *Solution by the* PROPOSER.

1. Mark off the figures into periods of 7 digits, making the unit figure the last one of a period.

2. Find the greatest seventh power in the first period, and subtract: bring down the second period, and put the root just found into the quotient.

3. Proceed to form a trial divisor as follows:—

A. Put down 7 times the sixth power of the quotient, and add 6 cyphers to it. Judge by division what the new figure of the quotient must be.

B. Square the quotient as now completed, and multiply it by 14 times the new figure and by the cube of the quotient minus the new figure.

C. Cube the quotient as now completed, and multiply it by 7 times the square of the new figure and by the quotient minus the new figure.

D. Add the fifth power of the new figure to seven times the fifth power of the quotient minus the new figure, and multiply by the new figure.

4. Add together the results of A, B, C, D, treat the sum as a divisor, multiply by the new figure, subtract and take down a new period, and proceed as before, paragraph 3, continuing the process until all or a sufficient number of figures are found.

[The foregoing solution gives what Mr. WHITWORTH states that he required; but the following solution has been sent by many correspondents.]

II. *Solution by* J. O'REGAN; E. RUTTER; *and others.*

Let N denote any number whose root is to be found, n the index of the root, a its nearest rational root, or a^n the nearest rational power to N, whether greater or less, x the remaining part of the root sought, which may be either positive or negative; then the required root is given by

$$N^{\frac{1}{n}} = \frac{(n+1)N+(n-1)a^n}{(n-1)N+(n-1)a^n}a;$$

that is, in words, to $n+1$ times N add $n-1$ times a^n, and to $n-1$ times N add $n+1$ times a^n, then the former sum multiplied by a and divided by the latter sum will give the root nearly.

By substituting for n in the general formula severally the numbers 2, 3, 4, 5, &c., we shall get the following particular expressions, as adapted to the 2nd, 3rd, 4th, 5th roots, &c. :—

$$\frac{3N+a^2}{N+3a^2}\cdot a = N^{\frac{1}{2}},\quad \frac{2N+a^3}{N+2a^3}\cdot a = N^{\frac{1}{3}},\quad \frac{5N+3a^4}{3N+5a^4}\cdot a = N^{\frac{1}{4}},$$

$$\frac{3N+2a^5}{2N+3a^5}\cdot a = N^{\frac{1}{5}},\quad \frac{7N+5a^6}{5N+7a^6}\cdot a = N^{\frac{1}{6}},\quad \frac{4N+3a^7}{3N+4a^7}\cdot a = N^{\frac{1}{7}}.$$

Example 1.—To find the square root of 365; $N = 365$, $n = 2$, the nearest square is 361, whose root is 19; hence we have

$$\frac{3N+a^2}{N+3a^2}\cdot a = \frac{1456}{1448}\cdot 19 = 19\cdot10497, \text{ \&c.}$$

To approach still nearer, substitute this last found root for a, the values of the other letters remaining as before; then we have

$$a^2 = \frac{19^2\times 182^2}{181^2} = \frac{3458^2}{181^2};\ \text{hence}\ \frac{3N^2+a^2}{N^2+3a^2}\cdot a = \frac{3458\times 47831059}{181\times 47831057},$$

which, being brought into decimals, gives the square root of 365 true to twenty places of decimals.

Example 2.—To find the seventh root of $126\cdot2$; we have $N = 126\cdot2$, $n = 7$, the nearest root $a = 2$, also $a^7 = 128$; hence $4N+3a^7 = 888\cdot8$, $3N+4a^7 = 890\cdot8$; therefore

$$\tfrac{8888}{4453} = 1\cdot995957 = \text{the root very exact by one operation.}$$

Example 3.—To find the 365th root of $1\cdot05$, or the amount of *one pound* for 1 day at 5 per cent. per annum compound interest. Here $N = 1\cdot05$, $n = 365$, $a = 1$ the nearest root; hence $366N+364a = 748\cdot3$, and $364N+366a = 748\cdot2$; hence $748\cdot2 : 748\cdot3 = 1 : \tfrac{7483}{7482} = 1\cdot00013366$, the root required, very exact, by *one operation.*

6208. (By E. W. Symons, M.A.)—Find the equation of the lines that join the origin to the points of intersection of the curves

$$u_1+u_2+\ldots+u_m = 0,\quad v_1+v_2+\ldots+v_n = 0.$$

Solution by W. J. C. Sharp, M.A.; W. Griffith; *and others.*

Form the eliminant of

$$z^{m-1}u_1+z^{m-2}u_2+\ldots+u_m = 0,\quad z^{n-1}v_1+z^{n-2}v_2+\ldots+v_n = 0;$$

and the result, which will be of constant weight mn, will be the equation

required. In the same way, eliminant of

$$z^m u_0 + z^{m-1} u_1 + \dots + u_m = 0, \quad z^n v_0 + z^{n-1} v_1 + \dots + v_n = 0$$

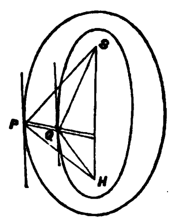
will be the equation to the lines, when the origin is not upon the curves.

If u_1, u_2, &c., v_1, v_2, &c., be ternary quantics in xyz, the same eliminant will give the equation to the cone standing upon the intersection of the surfaces.

If the same line pass through two intersections, this will be a double line on cone, and so on.

6620. (By G. F. Walker, M.A.)—Prove that the line of quickest descent, from one to the other of two confocal ellipses whose major axis is vertical, subtends equal angles at the foci.

Solution by A. Anderson, B.A., *and others.*



Let PQ be the line; $\angle PSH = S_1$, $\angle QSH = S_2$; then, since the normals are parallel, we have

$$S_1 + \tfrac{1}{2}P = S_2 + \tfrac{1}{2}Q, \text{ and } H_1 + \tfrac{1}{2}P = H_2 + \tfrac{1}{2}Q;$$

therefore $\qquad S_1 - H_1 = S_2 - H_2,$

and $\qquad S_1 - S_2 = H_1 - H_2,$

which proves the theorem. A concentric rectangular hyperbola can be drawn through PQSH.

4706. (By S. Tebay, B.A.)—A bag contains n tickets, marked with the numbers 1, 2, 3, ... n, representing prizes of the respective values a_1, a_2, a_3, ... a_n. A person draws m tickets, and selects the highest, the rest being replaced; a second person draws m tickets, and selects the highest, the rest being replaced; and so on for three or more persons. Find the expectations.

Solution by the Proposer.

Let $N_r = \dfrac{(n-r)!}{m!\,(n-m-r)!}$, $M_r = \dfrac{(m+r-1)!}{(m-1)!\,r!}$. The number m can be selected in one way, and the expectation is $\dfrac{1}{N} a_m$; $m+1$ can be selected in $m = M_1$ ways, and the expectation is $\dfrac{M_1}{N} a_{m+1}$; $m+2$ can be selected in $\dfrac{m(m+1)}{1.2} = M_2$ ways, and the expectation is $\dfrac{M_2}{N} a_{m+2}$; and so on. Hence the whole expectation is

$$\frac{1}{N}\left(a_m + M_1 a_{m+1} + M_2 a_{m+2} + \dots + M_{n-m} a_n\right).$$

The chance that m was selected is $\frac{1}{N}$, and the expectation, as before, is

$$\frac{1}{NN_1}(a_{m+1}+M_1 a_{m+2}+\dots+M_{n-m-1}a_n).$$

The chance that $m+1$ was selected is $\frac{M_1}{N}$, and the expectation is

$$\frac{1}{NN_1}(a_m+M_1 a_{m+2}+\dots+M_{n-m-1}a_n);$$

and similarly for $m+2$, $m+3$, &c.; therefore the whole expectation is

$$\frac{1}{NN_1}[M_1+M_2+\dots+M_{n-m-1}]\,a_m$$
$$+\frac{1}{NN_1}[1+M_1(M_2+M_3+\dots+M_{n-m-1})]\,a_{m+1}$$
$$+\frac{1}{NN_1}[M_1(1+M_1)+M_2(M_3+M_4+\dots+M_{n-m-1})]\,a_{m+2}$$
$$+\frac{1}{NN_1}[M_2(1+M_1+M_2)+M_3(M_4+M_5+\dots+M_{n-m-1})]\,a_{m+3}+\&c.$$

In like manner, by supposing every two of the numbers m, $m+1$, &c., to have been selected, we can find the expectation for the third drawing; and so on for the fourth, &c.

If a_m, a_{m+1}, &c., be equal, the expectation is a_m, and we have

$$\frac{n!}{m!\,(n-m)!}=1+m+\frac{m(m+1)}{1.2}+\dots+\frac{(n-1)!}{(m-1)!\,(n-m)!}.$$

4994. (By Professor SYLVESTER, F.R.S.)—The coordinates of a unicursal quartic x, y, z are given proportional to PQ', $P'Q$, QQ', where

$$P=a+bt+ct^2,\quad P'=a'+b't+c't^2,\quad Q=\alpha+\beta t+\gamma t^2,\quad Q'=\alpha'+\beta't+\gamma't^2.$$

Show that two of the nodes are the intersections of xy with z, and that the coordinates of the third node are in the proportion of $LM' : L'M : MM'$, where L, M, L', M' are the four first minors taken in a certain order of the rectangular matrix

$$\begin{vmatrix} a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix}$$

Solution by W. J. CURRAN SHARP, M.A.

$$\mu x = PQ' = (a+bt+ct^2)(\alpha'+\beta't+\gamma't^2),$$
$$\mu y = P'Q = (a'+b't+c't^2)(\alpha+\beta t+\gamma t^2),$$
$$\mu z = QQ' = (\alpha+\beta t+\gamma t^2)(\alpha'+\beta't+\gamma't^2).$$

Hence $x=0$, $z=0$ gives $Q'=0$; or t has two distinct values at this point, which is therefore a node; similarly $y=0$, $z=0$ is a node. At the third node the same values of t must come from

$$x(\alpha+\beta t+\gamma t^2)=z(a+bt+ct^2),\quad y(\alpha'+\beta't+r't^2)=z(a'+b't+c't^2);$$

therefore $$\frac{\gamma x - cz}{\gamma' x - c' z} = \frac{\beta x - bz}{\beta' y - b' z} = \frac{\alpha x - az}{\alpha' y - a' z};$$

$$\therefore \quad x \begin{vmatrix} a' \,.\, \alpha \,.\, \alpha' \\ b' \,.\, \beta \,.\, \beta' \\ c' \,.\, \gamma \,.\, \gamma' \end{vmatrix} = z \begin{vmatrix} \alpha' \,.\, a' \,.\, a \\ \beta' \,.\, b' \,.\, b \\ \gamma' \,.\, c' \,.\, c \end{vmatrix}, \quad y \begin{vmatrix} \alpha \,.\, \alpha' \,.\, a \\ \beta \,.\, \beta' \,.\, b \\ \gamma \,.\, \gamma' \,.\, c \end{vmatrix} = z \begin{vmatrix} \alpha \,.\, a' \,.\, a \\ \beta \,.\, b' \,.\, b \\ \gamma \,.\, c' \,.\, c \end{vmatrix}.$$

or $$x\text{M} = z\text{L} \text{ and } y\text{M}' = z\text{L}';$$

and therefore $$x : y : z = \text{LM}' : \text{L}'\text{M} : \text{MM}'.$$

6472. (By Professor TOWNSEND, F.R.S.)—Two normals to an ellipse being supposed to intersect at right angles on the curve; required the limits to their ratio requisite to their being both maxima vectors from their point of intersection to the curve.

Solution by the PROPOSER.

Denoting by r and s the lengths of the two normal chords, by p and q those of the two perpendiculars from the centre upon the tangents at their normal extremities, by ρ and σ the radii of the two circles of curvature at the same points, and by a and b the two semi-axes of the curve; then, the two normal chords determining manifestly, with the connector of their normal extremities, a triangle of maximum area inscribed to the ellipse, we have consequently $p = \frac{2}{3}r$, $q = \frac{2}{3}s$, and $pq = \frac{2}{9}(3\sqrt{3})\,ab$, and therefore $\rho = \dfrac{a^2b^2}{p^3} = \frac{1}{2}\dfrac{s^2}{r}$, and $\sigma = \dfrac{a^2b^2}{q^3} = \frac{1}{2}\dfrac{r^2}{s}$; hence, in order that ρ should be $< r$ and that σ should be $< s$, which are requisite in order that r and s should be both maxima vectors from their common extremity to the curve, we must have $s \div r$ and $r \div s$ both $< \sqrt{2}$; therefore, &c.

6677. (By Dr. MACFARLANE, F.R.S.E.)—(1) A lady, on being asked about a photograph in her album, gave the following answer:—"You know that I have no daughters; that person's daughter's son was the father of a grandchild of mine." [Also solve, by a like method, the two following problems:—

(2) "If Dick's father is Tom's son, what relation is Dick to Tom?"
(3) "Sisters and brothers I have none, but that man's father is my father's son."]

Solution by the PROPOSER.

1. I have proposed this question for the purpose of illustrating an extension of the Algebra of Logic which I have recently made in several papers contributed to the Royal Society of Edinburgh on an *Algebra of Relationship*.

Let A denote the lady, X the person referred to. Let c denote child, a subscript m denote male, and a subscript f denote female. Then c^{-1} is the proper expression for parent. Σ is used to denote *all*.

The first condition gives

$\Sigma_f c_f A = 0$, and the second $_m c_f X = {}_m c^{-1} c c_f A$..............(1, 2).

From (1) and (2), we have

$\Sigma_m c_f A = \Sigma c_f A$, $X = c^{-1}{}_f c^{-1}{}_m c^{-1} c c_f A = c^{-1}{}_f c^{-1}{}_m c^{-1} c_m c_f A$;

but $_m c^{-1} c_m = {}_m$ always, therefore $X = c^{-1}{}_f c^{-1}{}_m c_f A$; but $_f c^{-1} c_f = {}_f$ always, therefore $X = c^{-1}{}_f A$; that is, X was either the father or the mother of the lady.

The problems mentioned in the editorial note are solved as follows:—

2. Let D denote Dick and T denote Tom, then the given relation is expressed by the equation $_m c^{-1}{}_m D = {}_m c_m T$, that is, the male parent of the man D is identical with a male child of the man T. It is required to transform this equation so that D may be left by itself. The rule given in my papers on an *Algebra of Relationship* for such a transformation is as follows:—A symbol may be removed from the front of one side of an equation, provided its reciprocal be placed in front of the other side, the reciprocals of the sex-symbols m and f being m and f respectively.

Hence $D = {}_m c_m c\ T$; that is, Dick is a son of a son of the man Tom. We may drop one or more of the sex-symbols if we choose; thus $D = {}_m c^2 T$, that is, Dick is a grandson of Tom; or $D = c^2 T$, that is, Dick is a grandchild of Tom.

Suppose that we have given, in addition, that Bill's daughter *is* Tom's mother. What is the relation of Dick to Bill?

The equations are $_m c^{-1}{}_m D = {}_m c_m T$, $_f c_m B = {}_f c^{-1}{}_m T$.. ....(1, 2).

From (1) and (2), $D = {}_m c_m c_m T$, $T = {}_m c_f c_m B$, therefore $D = {}_m c_m c_m c_f c_m B$; that is, Dick is a son of a son of a son of a daughter of Bill. This involves that $D = {}_m c^4 B$, that is, Dick is a great-great-grandson of B; or $D = c^4 B$, that is, Dick is a great-great-grandchild of Bill.

3. Let A denote the speaker, and X the person referred to. Then the first clause gives us the equation $\Sigma c c^{-1} A = 2A$ ....(1),

that is, all the children of the parents of A is A; and the second clause gives the equation $_m c^{-1}{}_m X = {}_m c_m c^{-1} A$..............................(2).

From (1) it follows that $\Sigma c_m c^{-1} A = A$, therefore from (2) $_m c^{-1}{}_m X = {}_m A$, that is, A was the father of the man X; or $X = {}_m c_m A$, that is, X was a son of the man A.

6634. (By Elizabeth Blackwood.)—A straight line is divided into n random segments, show that the chance that these segments can form an n-sided polygon is $1 - n\,2^{1-n}$.

Solution by the Proposer.

Postulate:—To divide any magnitude into n random parts, take n random points in the circumference of any circle, and divide the given magnitude into n parts proportional to the n random arcs into which the circumference is thus divided.

Let x denote the statement, *Some one of the n random arcs into which the circumference is divided will be greater than the sum of the $n-1$ others*, that is, *greater than half the circumference.* We are required to find $1 - x_n$, in which x_n denotes *the chance that x is true.*

Measuring throughout in the *positive* direction of the circumference, le a denote the statement, (1) *Every one of the $n-1$ points* B, C, D, &c. *wil be less than half the circumference distant from* A; let b denote the statement, (2) *Every one of the $n-1$ points* A, C, D, &c. *will be less than half the circumference distant from* B; and so on for every point in succession. Then, evidently, $x = a+b+c+\dots$ Hence

$$x_s = (a+b+c+\dots)_s = a_s+b_s+c_s+\dots = na_s = n\,2^{1-n},$$

for only one of the n statements a, b, c, &c. can be true, and they are all equally probable. Thus, the required chance is $1-n\,2^{1-n}$.

6610. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—

To a wee planetoid, but recently out,
I am bound to dispatch an opinion,
On how to effect a design they're about
Of improving their little dominion.
Tired of their islands, they long for a Continent:
Here is the statement they give me, their want anent —

Their cities line their woody shore,
And creeks, that curl around in curious ways,
Beating the bays
Of Hellas or of Scotland;
So that of land is little more
About them than of not-land.

To no one city can there be
Of home approaches more than sacred three.
Each has two shore lines, and the only other
Is ferry o'er the creek to fronting town,
Or high-road, bound of province, to another,
Washed by another inlet. Peer and clown,
The laws from olden time forbidding byroad,
Must travel by that shore, or boat, or high-road.
Yet can two cities, ferry-bound,
Have commerce over ground
Along the winding shore-line round.

And there are towns on every strand
Without or ferry or a road inland,
Which, two and two on different isles, are able
To speak through ocean by electric cable.
These triad lines must be,
Unchanged for ever, bounds of earth and sea.

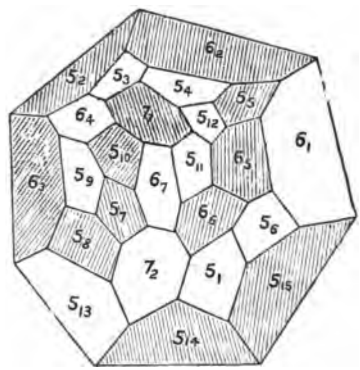
The ferries or the cable-lines between,
Some lordships of their shallow floods they mean
To bank and raise, the while they excavate
Along high road or roads, and inundate
What provinces they must, to make one strand,
And every cable a ferry, or edge of land.
How to contrive that every town may keep
Its own third line with piers along the deep?

To him who can tell them they give the remar-
Kable profit of purchasing shares at par;
And the Council will make him an F.S.R

On islands, three, four, or two,
Towns, to threescore or two,
Cover with triedral summits your n-edron;
When they are penned, run
Over your islands a pencilling cloud,
Giving the cities the shore-lines allowed.
The white will all be
Ferry, cable, and sea.
You may feel a bit proud,
If, after some labour, you find what you want, an ent-
Ire single circle of towns on a Continent.
When it is found, there is nothing to face
But proof of a rule to fit every case.

Solution by the Proposer.

The theorem—that on every p-edron P, having only triedral summits, a closed circle of edges passes once through every summit, has this provoking interest, that it mocks alike at doubt and proof.



Such a circle divides the face of P into a dark and a white sheet, so that no summit is unshared, *i.e.*, of three white or three dark faces, and so that there is neither a zone of white faces about a dark island, nor one of dark faces about a white lake, except the two sheets, each of which is a zone about the other. If s be the number of summits, and if A, B ... K be the m faces of the white sheet and Q, R ... V the $p-m$ dark ones, the equations following must be satisfied:

$$A+B+\ldots+K = s+2(m-1); \quad Q+R+\ldots+V = s+2(p-m-1) \ldots (F).$$

Every such selection of m faces of P that we make is a faultless white sheet, and the remaining faces a faultless dark one, if neither contains a zone nor a summit unshared.

In seeking this circle on any p-edron P, we first choose, from a promising case of equations (F), the principal faces of the white sheet, making the rest of such faces dark; we then add to both the faces dictated by the rule—no summits unshared. There is then a residue of pentagons and quadrilaterals to be so determined that neither sheet shall contain a zone nor an unshared summit. Of triangles, which can add no difficulty, I need not speak. The white zones which we run the risk of making are such as contain only white faces selected; and the like of the dark ones.

The faces collateral with the A-gon A are the zone about A; the zone about two collaterals AB is the sum of their zones, omitting A and B and repetitions. All zones are given to inspection of P, and no zone can have a summit unshared. Successes have been easy and plentiful on every P yet examined.

If P be our figure, $s = 46$, and the $p\ (= 25)$ faces are $7^3 6^7 5^{15}$, where 7^3 means 777, &c. The only possible selections of white and dark sheets t

satisfy equations (F), making the base 7-gon white, and writing dark over white, are

$$\frac{6^2 5^{12}}{7^3 6^6 5^3},\quad \frac{6^5 5^8}{7^3 6^2 5^7},\quad \frac{76^3 5^9}{7^2 6^4 5^6},\quad \frac{75^{13}}{7^2 6^7 5^2},\quad \frac{76^6 5^5}{7^2 65^{10}},\quad \frac{7^2 65^{10}}{76^6 5^5},\quad \frac{7^2 6^7 5^2}{75^{13}},\quad \frac{7^2 6^4 5^6}{76^3 5^9}.$$

Sheets of all these forms will be found on some 25-edra, having the signature $7^3 6^7 5^{15}$, but not always on one P.

Starting with our cut as datum, we have two white zones about two islands $\dfrac{5_{14}5_{15}}{7_1 7_2 5_1 5_6 5_{13} 6_1}$ and $\dfrac{7_3 6_2 6_3 6_5 6_6 5_2 5_7 5_8 5_{10} 5_5}{7_1 7_2 6_1 6_4 6_7 5_1 5_3 5_4 5_6 5_9 5_{11} 5_{12} 5_{13}}$, zones which have six white faces in common, and can both be destroyed by darkening any one of the six. We may darken 5_{13}, thus making one sheet of land and one of sea, both without zones. Our sheets are

$$\frac{7_3 6_2 6_3 6_5 6_6 5_2 5_5 5_7 5_8 5_{10} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_7 5_1 5_3 5_4 5_6 5_9 5_{11} 5_{12}} \text{ of the form } \frac{76^4 5^8}{7^2 6^3 5^7},$$

which does not satisfy equations (F). But we see that, by making white any dark 6-gon, and darkening any white 5-gon, we can obtain the third of the above written forms. We cannot make 6_2 or 6_6 white without the unshared summit $7_1 6_1 6_2$ or $7_2 6_6 6_7$. Make 6_5 white, and 5_6 dark, to avoid the unshared summit $6_5 6_1 5_6$. We have now two sheets

$$\frac{7_3 6_2 6_3 6_6 5_2 5_5 5_6 5_7 5_8 5_{10} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_5 6_7 5_1 5_3 5_4 5_9 5_{11} 5_{12}},$$

numerically correct and containing no zone; but tactically wrong by having the unshared summits, $6_3 5_8 5_{13}$ all dark, and $6_5 5_{11} 5_{12}$ all white

Exchange 5_8 and 5_{11}; then $\dfrac{7_3 6_2 6_3 6_6 5_2 5_5 5_6 5_7 5_{10} 5_{11} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_5 6_7 5_1 5_3 5_4 5_8 5_9 5_{12}}$ is numerically and tactically a faultless pair, and if we exchange 5_8 instead of $5_8 5_{11}$ $\dfrac{7_3 6_2 6_3 6_6 5_2 5_5 5_6 5_7 5_{10} 5_{12} 5_{13} 5_{14} 5_{15}}{7_1 7_2 6_1 6_4 6_5 6_7 5_1 5_3 5_4 5_8 5_9 5_{11}}$ is another.

If the reader shades the dark sheet of either, he will see the convolutions of the white one.

Of the form $7^2 6^4 5^6$ we can select $3 . 35 . 5005 = 525525$ different white sheets numerically correct. We have just proved that two at least of this number give a faultless pair; and there are more than two, as given below. Let it be given that on every p-edron P having only triedral summits, this circle can be found. How far does this go towards proof that such a circle can be found on every such $(p+1)$-edron Q? Q becomes a P by erasure of an edge, and P becomes Q by drawing that edge.

Draw in our given P an edge bisecting 6_1 and making the new summits 855 and 755. The signature of this Q is $87^3 6^6 5^{17}$. Keeping the base 8-gon white, the possible sheets, by equations (F), will be of the forms

$$\frac{6^4 5^{10}}{87^3 65^7},\quad \frac{76^5 5^7}{87^2 5^{10}},\quad \frac{76^2 5^{11}}{87^2 6^3 5^6},\quad \frac{7^2 5^{12}}{876^5 5^5},\quad \frac{7^2 6^3 5^8}{876^2 5^9},\quad \frac{7^3 65^9}{86^4 5^8},\quad \frac{7^3 6^4 5^5}{865^{12}}.$$

The entire number of selections of every form of sheets on P is 1320396; that full number on Q is 1430618, or 110222 more. Our added risks of error on Q to those on P are the two new summits (755) and (855) to be excluded if unshared from our sheets, and the two new zones about the new 5-gons. Every other zone on Q is also a zone on P.

That is, we have in our power 110222 more selections on Q than on P; and only four more risks of error. Ergo, say I, it is impossible to doubt that we shall find on Q more faultless sheets than on P.

This is the only attempt to prove the general theorem that I desire now to offer. Demonstration it is not; but it will probably carry conviction until a p-edron is produced, on which the sought circle cannot be found. A rigorous demonstration is much to be desired, flowing from the simple definition of a p-edron with triedral summits. But I share the opinion of Professor P. G. TAIT, that our prospect of obtaining such demonstration is very remote indeed. It is my impression that he knows more about these circles than any other.

Have we here a glimpse of the Tactic Calculus which will one day make short work of questions like this? Is it impossible that the following theorem may appear on the pillars of Science? Let T_n be a given tactical success defined by the integer n, and let T_{n-1}, T_{n-2}, &c., be all given certainties. If it can be proved that the chances against the success T_{n+1} diminish without limit as n increases, T_n is to be considered a certainty for every integer n. I do not here endorse this theorem.

When the summits of a p-edron P are not all triedral, the equations (F) will be satisfied by white and dark sheets whose common circuit is a circle through all the summits; and if they are satisfied so that there is no zone, and so that every summit is shared in two contiguous white and contiguous dark portions, the required circle is found.

The equations (F) are impossible when the number s of summits is odd, while the faces are all even-angled. Such cases are easily made by crowning at m of its summits, by an m-ace making m even-angled faces, the $2r$-gonal face of a prism, completing a solid of $4r+1$ summits with even faces.

I find on our figured 25-edron one circle of the first form $\frac{6^2 5^{12}}{7^3 6^5 5^3}$, 17 of the form $\frac{6^5 5^8}{7^3 6 \cdot 5^7}$, 8 of the form $\frac{7 6^3 5^9}{7 \cdot 6^4 5^6}$, 4 of the form $\frac{7^2 6 5^{10}}{7 6^6 5^5}$, and 14 of the form $\frac{7^2 6^4 5^6}{7 6^3 5^9}$; in all 44, in which the base 7-gon is white. As in all of them the dark and white sheets can be exchanged, our puzzled planetarians have the choice of at least 88 different continents.

Attention to these circles was perhaps first called by me in *Phil. Trans.*, Vol. 148, 1858, page 160, from which the late Royal Astronomer, Sir W. R. HAMILTON, took his idea of his "Icosian Game," as he informed me, when he did me the honour to present me with his handsomest copy of the puzzle.

If neither confuting case nor rigorous demonstration of our theorem can be found, the proper thing will be to establish a conservation. Conservations are stamps of certainty which science prints on propositions about the future, which she is utterly unable to prove, but which she is led to believe by all experience on record. Such is the primary conservation which NEWTON so wisely formulated with the deliberate anthropomorphism *perseverare*, and which it is now the fashion to demonstrate scientifically by the grand law of the eternal standstill of motion. See SPENCER's mistranslation of NEWTON, "must persevere," F. Pr. 1880, p. 182. Such are all our conservations, down to that wise upstoring some forty years ago, and its most logical consequence the present conservation, of what is poetically called the potential energy of your grandfather's chimney-pots. They are useful disguises, arithmetically harmless, of our scientific ignorance and incompetence to pronounce upon events of the future.

Why should not Tactic have its Conservations, as well as the good old Dynamic and its modern false *propylæum* Kinetic? (False in division and

in naming; first, because there is no force not kinetical; and secondly, because there is no motion not dynamical. *Vide* NEWTON'S *VIS inertiæ.* To NEWTON they appeal; to NEWTON let them go.) And why should not I be immortal for manufacturing the first? I mean to make it a dead certainty, sealed in orthodox fashion with—must be so; I know—that whenever in the 24th century these circles are looked for on the 10^{10}-edra, they will be found. For me it is quite easy—thus: There is in the essential nature of all p-edra having only triedral summits a Tactical Conservation by cyclical correlation and coordination, of perichoretical co-inter-environment and interconfiguration, due to the conformativities and circularitivities of continuous and discontinuous differentiations and integrations which we Unfathomables perfectly understand. "That is an ultimate of ultimates." Let no fathomable touch it. I made it, because I will have the applause of the gifted minds whose superior "idiosyncrasies" (*vide* Appendix to SPENCER'S *First Principles*, edition of 1880) enable them at sight exactly "to frame the ideas answering to" any of these plumed horse-marines of philosophy, "highly abstract words," even to the words of the very highest degree of abstractness "in which," after the pattern of Tactical Conservation, "Evolution at large is expressed."

Does the curious reader ask why I call "highly abstract words" horse-marines? Simply, because there are no horse-marines outside the brains of unfathomable Profundity. Is *negro* a highly black word? Is *change* or *vicissitudinality* a highly abstract one? The general idea is abstract enough.

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$$\phi x = c+\frac{\log[(a-1)x+b]}{\log a}, \quad \phi x = c+\frac{\log\log x}{\log r},$$

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